

BAR CATEGORY OF MODULES AND HOMOTOPY ADJUNCTION FOR TENSOR FUNCTORS

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ABSTRACT. Given a DG-category \mathcal{A} we introduce the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. It is a DG-enhancement of the derived category $D(\mathcal{A})$ of \mathcal{A} which is isomorphic to the category of DG \mathcal{A} -modules with A_∞ -morphisms between them. However, it is defined intrinsically in the language of DG-categories and requires no complex machinery or sign conventions of A_∞ -categories. We define for these bar categories Tensor and Hom bifunctors, dualisation functors, and a convolution of twisted complexes. The intended application is to working with DG-bimodules as enhancements of exact functors between triangulated categories. As a demonstration we develop homotopy adjunction theory for tensor functors between derived categories of DG-categories.

1. INTRODUCTION

The machinery of triangulated categories as originally developed by Verdier in [Ver96] has long been seen as imperfect. In recent years it became increasingly popular to fix the technical issues arising from these imperfections by working formally in a DG-enhancement. Given a triangulated category \mathcal{T} its DG-enhancement is a differentially graded (DG) category \mathcal{A} with $H^0(\mathcal{A}) \simeq \mathcal{T}$. We recommend [Kel94], [Toë11], [AL13, §2] for an introduction to DG-categories, [LO10], [AL13, §4] for an introduction to DG-enhancements, and [Toë07] for the underlying technical framework.

Most triangulated categories which arise in algebra and geometry are derived categories and thus are naturally $H^0(-)$ truncations of DG-categories of complexes of objects in an abelian category. In other words, they possess a natural DG-enhancement. Examples include the derived categories of sheaves or constructible sheaves on a topological space or of quasi-coherent sheaves, coherent sheaves, or D -modules on an algebraic variety. Moreover, in a number of these examples (e.g. for the derived category of any Grothendieck category) it was established that this natural enhancement is unique up to quasi-equivalence [LO10], [CS15]. This means, roughly, that functorial constructions carried out in a DG-enhancement will produce the same results in the triangulated category regardless of the choice of the enhancement. Finally, for technical reasons it is best to work in Morita framework where instead of enhancing \mathcal{T} with some DG-category \mathcal{A} we enhance \mathcal{T} with the DG-category of modules over some DG-category \mathcal{A} .

Let \mathcal{T} and \mathcal{U} be two triangulated categories. One of the aforementioned major issues is that the exact functors $\mathcal{T} \rightarrow \mathcal{U}$ do not themselves form a triangulated category. However, if we DG-enhance \mathcal{T} and \mathcal{U} by DG-categories \mathcal{A} and \mathcal{B} then it follows from a fundamental result of Toën [Toë07] that DG-enhanceable functors form an exact category which can be identified with the derived category of \mathcal{B} -perfect \mathcal{A} - \mathcal{B} -bimodules. In algebraic geometry, if $\mathcal{T} = D(X)$ and $\mathcal{U} = D(Y)$, the DG-category of (perfect) \mathcal{A} - \mathcal{B} -bimodules is a natural DG-enhancement of the product $X \times Y$ and the above translates to DG-enhanceable functors being precisely the Fourier-Mukai transforms [Toë07, §8], [LS16], [AL13, §4.3]. Thus studying \mathcal{A} - \mathcal{B} -bimodules as DG-enhancements of exact functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ can be considered as the universal Fourier-Mukai theory for enhanced triangulated categories.

In their work on spherical functors [AL13] and a range of related categorification problems the authors of this paper found it necessary to adopt the above approach to carry out various technical constructions. These included taking cones of natural transformations, in particular of units and counits of adjunction, and, more generally, convolutions of complexes of functors. Even though working in quasi-equivalent enhancements produces the same results, we have discovered that from the point of view of actually carrying out explicit computations choosing a suitable enhancement makes a world of difference. In this paper we would like to propose and study the DG-enhancement framework for the derived categories of DG-modules and bimodules which we eventually found to be most suitable for working with enhancements of exact functors. As a demonstration we write down homotopy adjunction theory for bimodules in this framework.

Let \mathcal{A} be a DG-category and let $\mathbf{Mod}\text{-}\mathcal{A}$ be the DG-category of \mathcal{A} -modules. There are two enhancements commonly used in the literature for $D(\mathcal{A})$: the full subcategory $\mathcal{P}(\mathcal{A})$ of the h -projective modules in $\mathbf{Mod}\text{-}\mathcal{A}$, and the Drinfeld quotient $\mathbf{Mod}\text{-}\mathcal{A}/\mathcal{Ac}(\mathcal{A})$ where $\mathcal{Ac}(\mathcal{A})$ is the full subcategory of acyclic modules [Dri04]. Neither turned out to be suitable for our purposes. The problem with the Drinfeld quotient is that its

morphisms are inconvenient to work with explicitly. The problem with $\mathcal{P}(\mathcal{A})$ becomes apparent when working with bimodules. The diagonal bimodule \mathcal{A} , which corresponds to the identity functor $D(\mathcal{A}) \rightarrow D(\mathcal{A})$, is not h -projective. Hence every construction involving the identity functor has to be h -projectively resolved leading to many formulas becoming vastly more complicated than they should be, cf. [AL13].

This can be fixed by working with DG \mathcal{A} -modules and A_∞ -morphisms between them. In other words, the full subcategory $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ of the DG-category $\mathbf{Nod}_\infty \mathcal{A}$ of A_∞ \mathcal{A} -modules which consists of DG-modules. In $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ all quasi-isomorphisms are already homotopy equivalences, thus there is no need to take resolutions and we directly have $D(\mathcal{A}) \simeq H^0((\mathbf{Nod}_\infty \mathcal{A})_{dg})$. However, the machinery of A_∞ -categories lends itself poorly to explicit computations and formulas, in part due to the complicated sign conventions involved. It seems wasteful to bring in the full generality of A_∞ -language to only consider A_∞ -morphisms between DG-modules over a DG-category. To this end we introduce the *bar category of modules* over \mathcal{A} which can be regarded as a technical tool which simplifies the A_∞ -machinery involved to the extent which is actually necessary. It is a category which is *isomorphic* to $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$, yet it has an intrinsic definition entirely in the language of DG-modules and the bar complex $\bar{\mathcal{A}}$. Indeed, it is not necessary to know anything about A_∞ -categories to work with bar category of modules:

Definition (Definition 4.1). Let \mathcal{A} be a DG-category. Define the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ as follows:

- The object set of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is the same as that of $\mathbf{Mod}\text{-}\mathcal{A}$: DG-modules over \mathcal{A} .
- For any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ set

$$\mathrm{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F) = \mathrm{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F)$$

and write $\overline{\mathrm{Hom}}_{\mathcal{A}}(E, F)$ to denote this Hom-complex.

- For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ set $\mathrm{Id}_E \in \overline{\mathrm{Hom}}_{\mathcal{A}}(E, E)$ to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mathrm{Id} \otimes \tau} E \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\sim} E$$

where $\bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}$ is the canonical projection.

- For any $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ define the composition map

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(F, G) \otimes_k \overline{\mathrm{Hom}}_{\mathcal{A}}(E, F) \longrightarrow \overline{\mathrm{Hom}}_{\mathcal{A}}(E, G)$$

by setting for any $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha} F$ and $F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G$ the composition of the corresponding elements to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mathrm{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \mathrm{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G.$$

In Prop. 4.3 we show that $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is isomorphic to $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$. In particular, $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is indeed a DG-enhancement of the triangulated category $D(\mathcal{A})$. Let \mathcal{B} be another DG category. We define the bar category $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ of $\mathcal{A}\text{-}\mathcal{B}$ bimodules analogously (Defn. 4.6). We then define explicitly a pair of DG bifunctors $(-) \otimes_{\mathcal{B}} (-)$ and $\overline{\mathrm{Hom}}_{\mathcal{B}}(-, -)$ (Defns. 4.7 and 4.8) which are DG-enhancements of the derived functors $(-) \otimes_{\mathcal{B}} (-)$ and $\mathbf{R} \mathrm{Hom}_{\mathcal{B}}(-, -)$. We prove the Tensor-Hom adjunction for $\otimes_{\mathcal{B}}$ and $\overline{\mathrm{Hom}}_{\mathcal{B}}$ as functors between bar categories of bimodules and give formulas for its adjunction units and counits (Prop. 4.12). Finally, the aforementioned category isomorphism $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \simeq (\mathcal{A}\text{-}\mathbf{Nod}_\infty \mathcal{B})_{dg}$ identifies $\otimes_{\mathcal{B}}$ and $\overline{\mathrm{Hom}}_{\mathcal{B}}$ with their A_∞ -counterparts $\overset{\mathrm{L}}{\otimes}_{\mathcal{B}}$ and $\overset{\infty}{\mathrm{Hom}}_{\mathcal{B}}$ (Prop. 4.10).

We next set up the dualisation theory: we define functors $(-)^{\bar{\mathcal{A}}} = \overline{\mathrm{Hom}}_{\mathcal{A}}(-, \mathcal{A})$ and $(-)^{\bar{\mathcal{B}}} = \overline{\mathrm{Hom}}_{\mathcal{B}}(-, \mathcal{B})$ from $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ to $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and show them to be equivalences on the subcategories of \mathcal{A} - and \mathcal{B} -perfect bimodules, respectively (Lemma 4.28). Moreover, we show that for any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ the bimodules $M^{\bar{\mathcal{A}}}$ and $M^{\bar{\mathcal{B}}}$ enhance the derived functors $\mathbf{R} \mathrm{Hom}_{\mathcal{A}}(M, -)$ and $\mathbf{R} \mathrm{Hom}_{\mathcal{B}}(M, -)$ whenever M is \mathcal{A} - and \mathcal{B} -perfect, respectively (Lemma 4.30). This becomes crucial for the homotopy adjunction theory we develop later on.

The technical constructions mentioned above, such as cones of morphisms of functors and convolutions of complexes of functors, are usually carried out in the language of twisted complexes [BK90], [AL13, §3]. This requires the DG-enhancement to be pretriangulated or, ideally, strongly pretriangulated. Unlike the ordinary categories of modules, bar categories of modules are not strongly pretriangulated. They are, however, pretriangulated, and we write down the convolution functor $\mathrm{Pre}\text{-}\mathrm{Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}) \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ which is a quasi-equivalence and a homotopy inverse to the inclusion $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \hookrightarrow \mathrm{Pre}\text{-}\mathrm{Tr}(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$ (Defn. 4.36 and Cor. 4.38). We then write down explicit formulas for Tensor, Hom and duals of twisted complexes of bimodules which are necessary for the aforementioned computations (Lemmas 4.39 and 4.40).

For any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ we have the natural map $\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\alpha} M$ which is an analogue of the \mathcal{A} -action map $\mathcal{A} \otimes_{\mathcal{A}} M \rightarrow M$. The biggest drawback of bar categories is that this map is no longer an isomorphism, but only

a homotopy equivalence. However, we make a full study of the higher homotopies involved. Quite generally, as explained in [Dri04, §3.7], [Tab05], [AL13, Appendix A], in any DG-category any homotopy equivalence $y \xrightarrow{\alpha} x$ can be completed to the following system of morphisms and relations:

$$\begin{aligned}
 d\theta_x &= \alpha \circ \beta - \text{Id}_x, \\
 d\theta_y &= \text{Id}_y - \beta \circ \alpha, \\
 d\alpha &= d\beta = 0, \\
 d\phi &= -\beta \circ \theta_x - \theta_y \circ \beta.
 \end{aligned}
 \tag{1.1}$$

However, in our situation we can do better. We write down a homotopy inverse $M \xrightarrow{\beta_0} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M$ and a degree -1 endomorphism θ of $\mathcal{A} \overline{\otimes}_{\mathcal{A}} M$ for which we prove:

Proposition (Prop. 4.22). *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. The sub-DG-category of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ generated by α, β_0 and θ is the free DG-category generated by these modulo the following relations:*

$$\begin{array}{l}
d\alpha = d\beta_0 = 0, \\
d\theta = \text{Id} - \beta_0 \circ \alpha, \\
0 = \alpha \circ \beta_0 - \text{Id}, \\
\alpha \circ \theta = 0.
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc}
M & \begin{array}{c} \xrightarrow{\beta_0} \\ \xleftarrow{\alpha} \end{array} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \\
& & \begin{array}{c} \text{---} \theta \text{---} \\ \text{---} \end{array}
\end{array}
\end{array}
\quad (1.2)$$

Note that the relations in (1.2) can be obtained from those in (1.1) by demanding further that $\theta_x = 0$, $\alpha \circ \theta_y = 0$, and $\phi = \theta_y^2 \circ \beta$. We also prove analogous results for the homotopy equivalence $M \xrightarrow{\gamma} \overline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, M)$ which is the right adjoint of $\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\alpha} M$ (Prop. 4.25).

As an application of the above results, we use the bar category of bimodules to write down homotopy adjunction theory for DG \mathcal{A} - \mathcal{B} -bimodules as enhancements of tensor functors. An exact functor $D(\mathcal{A}) \xrightarrow{f} D(\mathcal{B})$ is a *tensor functor* if it is isomorphic to tensor multiplication by an \mathcal{A} - \mathcal{B} bimodule. This is equivalent to f being *continuous* (commuting with infinite direct sums) or to f having a right adjoint r (Prop. 5.1). We prove similar characterisations of r being continuous and of f having a left adjoint l (which is then automatically continuous) in terms of the corresponding bimodule being \mathcal{B} - or \mathcal{A} -perfect (Prop. 5.2). Finally, if we have an adjoint triple (l, f, r) of tensor functors we write down homotopy adjunction for their enhancements. If $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ enhances f , we show that $M^{\tilde{\mathcal{A}}}$ and $\mathcal{M}^{\tilde{\mathcal{B}}}$ enhance l and r , respectively. The corresponding functors $(-)\overline{\otimes}_{\mathcal{A}}M$, $(-)\overline{\otimes}_{\mathcal{B}}M^{\tilde{\mathcal{B}}}$, $(-)\overline{\otimes}_{\mathcal{B}}M^{\tilde{\mathcal{A}}}$ between $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ are then homotopy adjoint. We write down their homotopy units and counits (Defns. 5.3 and 5.5) and then use the results from Section 4.3 to write down the higher homotopies by which they differ from genuine adjunctions (Defn. 5.6 and Prop. 5.7).

Briefly, on the layout of this paper. In Section 2 we give prerequisites on DG-categories. We also prove a minor new result, the Rectangle Lemma 2.1, from which it follows that a one-sided morphism (f_{ij}) of twisted complexes of DG-modules induces a quasi-isomorphism of their convolutions if each f_{ii} is a quasi-isomorphism.

In Section 3 we likewise give prerequisites on A_∞ -categories, modules, and bimodules. There are also minor new results: in §3.4 we define Tensor and Hom bifunctors for A_∞ -bimodules. Previously, to our knowledge, these were only written down for A_∞ -modules [LH03, §4], [Kel01, §6]. In §3.5 we identify the functor $(-)\overset{\infty}{\otimes}_{\mathcal{A}}\mathcal{A}$ as the functorial semifree resolution for $(\mathbf{Mod}_\infty\mathcal{A})_{hu}$. This clarifies several results in [LH03], *Chapitre 4*, e.g. the proof that every module whose bar construction is acyclic is homologically unital.

In Section 4 we introduce the bar categories of modules and bimodules along with all the supplementary constructions described above. Finally, in Section 5 we construct homotopy adjunction for tensor functors.

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2. PRELIMINARIES ON DG-CATEGORIES

For a brief introduction to DG-categories, DG-modules, and the technical notions involved we direct the reader to [AL13], §2-4. The present paper was written with that survey in mind. We employ freely any

notion or piece of notation introduced in [AL13], §2-4. We particularly stress the importance of the material on *twisted complexes* in [AL13], §3.

However, for the convenience of the reader, below we briefly summarise some of the most relevant facts and notation.

Let \mathcal{A} be a DG-category. The *derived category* $D(\mathcal{A})$ of \mathcal{A} is the localisation of the homotopy category $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ of (right) DG-modules by the class of quasi-isomorphisms. It is constructed as the Verdier quotient of $H^0(\mathbf{Mod}\text{-}\mathcal{A})$ by the full subcategory $H^0(\mathcal{Ac}(\mathcal{A}))$ consisting of acyclic modules. It comes, in particular, with the canonical projection $H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A})$. See [AL13, §2.1.6] for further detail.

Let \mathcal{B} be a DG-category and $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ be an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule. Then we have canonical maps

$$\mathcal{A} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(M, M) \quad (2.1)$$

$$\mathcal{B} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{A}}(M, M) \quad (2.2)$$

in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, respectively. These are called \mathcal{A} - and \mathcal{B} -*action maps*, respectively, because they represent the action of \mathcal{A} (resp. \mathcal{B}) on M by \mathcal{B} -module (resp. \mathcal{A} -module) morphisms. Note that e.g. bimodule $\text{Hom}_{\mathcal{B}}(M, M)$ has an \mathcal{A} -algebra structure defined by the composition. It therefore defines a DG-category with the same set of objects as \mathcal{A} . This DG-category is precisely the image of the functor $\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{B}$ which corresponds to M , cf. [AL13, §2.1.5].

Let \mathcal{C} and \mathcal{D} be DG-categories. For any $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, $L \in \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, and $N \in \mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ we have the *composition* map in $\mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$

$$\text{Hom}_{\mathcal{B}}(M, N) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(L, M) \xrightarrow{\text{cmps}} \text{Hom}_{\mathcal{B}}(L, N)$$

which is defined levelwise by composition in $\mathbf{Mod}\text{-}\mathcal{B}$.

Let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. We have the usual Tensor-Hom adjunction: for any DG-category \mathcal{C}

$$(-) \otimes_{\mathcal{A}} M : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$$

is left adjoint to

$$\text{Hom}_{\mathcal{B}}(M, -) : \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}.$$

The adjunction counit

$$\text{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} M \xrightarrow{\text{ev}} \text{Id} \quad (2.3)$$

is called the *evaluation map*, as it is defined by

$$\alpha \otimes m \mapsto \alpha(m).$$

Similarly, we call the adjunction unit

$$\text{Id} \xrightarrow{\text{mlt}} \text{Hom}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{A}} M) \quad (2.4)$$

the *tensor multiplication map*, as it is defined by

$$s \mapsto s \otimes (-).$$

Analogously,

$$M \otimes_{\mathcal{B}} (-) : \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \rightarrow \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$$

is left adjoint to

$$\text{Hom}_{\mathcal{A}^{\text{opp}}}(M, -) : \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \rightarrow \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$$

with the adjunction counit

$$\begin{aligned} M \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, -) &\xrightarrow{\text{ev}} \text{Id} \\ m \otimes \alpha &\mapsto (-1)^{\deg(m) \deg(\alpha)} \alpha(m) \end{aligned} \quad (2.5)$$

and the adjunction unit

$$\begin{aligned} \text{Id} &\xrightarrow{\text{mlt}} \text{Hom}_{\mathcal{A}^{\text{opp}}}(M, M \otimes_{\mathcal{B}} (-)) \\ s &\mapsto (-1)^{\deg(-) \deg(s)} (-) \otimes s. \end{aligned} \quad (2.6)$$

Let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. The action of \mathcal{A} on M defines the canonical isomorphism

$$\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\sim} M \quad (2.7)$$

$$a \otimes m \mapsto a.m. \quad (2.8)$$

The right adjoint of (2.7) with respect to $(-) \otimes_{\mathcal{A}} M$ is the \mathcal{A} -action map $\mathcal{A} \xrightarrow{\text{act}} \text{Hom}_{\mathcal{B}}(M, M)$. The right adjoint of (2.7) with respect to $\mathcal{A} \otimes_{\mathcal{A}} (-)$ is the canonical isomorphism

$$M \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \quad (2.9)$$

$$m \mapsto (-1)^{\deg(-)\deg(m)}(-).m. \quad (2.10)$$

Similarly, we have canonical isomorphisms

$$M \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{\sim} M, \quad (2.11)$$

$$M \xrightarrow{\sim} \text{Hom}_{\mathcal{B}}(\mathcal{B}, M). \quad (2.12)$$

The canonical isomorphisms (2.9) and (2.12) identify evaluation maps with composition maps. For example,

$$\text{Hom}_{\mathcal{B}}(M, E) \otimes_{\mathcal{A}} M \xrightarrow{\text{Id} \otimes (2.12)} \text{Hom}_{\mathcal{B}}(M, E) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(\mathcal{B}, M) \xrightarrow{\text{cmps}} \text{Hom}_{\mathcal{B}}(\mathcal{B}, E) \xrightarrow{(2.12)^{-1}} E$$

is the evaluation map (2.3).

Finally, for all $M \in \mathcal{A}\text{-Mod-}\mathcal{B}$, $N \in \mathcal{D}\text{-Mod-}\mathcal{B}$ and $L \in \mathcal{C}\text{-Mod-}\mathcal{A}$ we have a canonical map

$$\begin{aligned} L \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(N, M) &\longrightarrow \text{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M) \\ l \otimes \alpha &\mapsto l \otimes \alpha(-) \end{aligned} \quad (2.13)$$

which is a quasi-isomorphism when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect, cf. [AL13, §2.2]. We can also write (2.13) as the composition

$$L \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\text{mlt} \otimes \text{Id}} \text{Hom}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\text{cmps}} \text{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M).$$

2.1. Rectangle Lemma. We will also need the following useful fact. Let \mathcal{A} be a DG-category. All twisted complexes in this section are considered to be over \mathcal{A} . We say that a map (f_{ij}) of twisted complexes is *one-sided* if $f_{ij} = 0$ for any $j < i$.

Lemma 2.1. *Let $E = (E_i, \alpha_{ij})$ and $F = (F_i, \beta_{ij})$ be one-sided twisted complexes. Let $f = (f_{ij})$ be a one-sided closed map $E \rightarrow F$ of degree 0.*

There exists a twisted complex $G = (G_i, \gamma_{ij})$ over $\text{Pre-Tr } \mathcal{A}$ with each

$$G_i = \left(E_i \xrightarrow[\deg.0]{(-1)^i f_{ii}} F_i \right)$$

such that

$$\text{Tot}(G_i, \gamma_{ij}) \simeq \text{Tot} \left(E \xrightarrow[\deg.0]{f} F \right)$$

in $\text{Pre-Tr } \mathcal{A}$.

Proof. Since f is closed of degree 0 and one-sided, we have $(df)_{ii} = df_{ii} = 0$, and

$$(df)_{ij} = (-1)^j df_{ij} + \sum_{k=i}^{j-1} \beta_{kj} f_{ik} - \sum_{k=i+1}^j f_{kj} \alpha_{ik} = 0 \quad j > i.$$

Define the twisted differentials $G_i \xrightarrow{\gamma_{ij}} G_j$ by the following diagram:

$$\begin{array}{ccc} E_i & \xrightarrow{(-1)^i f_{ii}} & F_i \\ \downarrow -\alpha_{ij} & \searrow f_{ij} & \downarrow \beta_{ij} \\ E_j & \xrightarrow{(-1)^j f_{jj}} & F_j \\ & & \deg.0 \end{array}$$

The degree of this map is $i - j + 1$. Note also that

$$\begin{array}{ccc} E_i & \xrightarrow{(-1)^i f_{ii}} & F_i \\ \downarrow d\alpha_{ij} & \searrow df_{ij} - (-1)^j f_{jj} \alpha_{ij} - (-1)^{i-j+1} (-1)^i \beta_{ij} f_{ii} & \downarrow d\beta_{ij} \\ E_j & \xrightarrow{(-1)^j f_{jj}} & F_j \end{array}$$

We claim that $G = (G_i, \gamma_{ij})$ is a twisted complex. For this we need to have for all i and j

$$(-1)^j d\gamma_{ij} + \sum_{k=i+1}^{j-1} \gamma_{kj} \gamma_{ik} = 0.$$

The map on the LHS has three components: $E_i \rightarrow E_j$, $F_i \rightarrow F_j$ and $E_i \rightarrow F_j$. The first two components vanish since E and F are twisted complexes. Computing the $E_i \rightarrow F_j$ component we get

$$\begin{aligned} (-1)^j (df_{ij} - (-1)^j f_{jj} \alpha_{ij} + (-1)^j \beta_{ij} f_{ii}) + \sum_{k=i+1}^{j-1} \beta_{kj} f_{ik} - \sum_{k=i+1}^{j-1} f_{kj} \alpha_{ik} = \\ = (-1)^j df_{ij} + \beta_{ij} f_{ii} + \sum_{k=i+1}^{j-1} \beta_{kj} f_{ik} - f_{jj} \alpha_{ij} - \sum_{k=i+1}^{j-1} f_{kj} \alpha_{ik} = (df)_{ij} = 0. \end{aligned}$$

Thus (G_i, γ_{ij}) is a twisted complex. Its total complex and that of $(E \xrightarrow{f} F)$ both have the i -th term

$$E_{i+1} \oplus F_i$$

and the ij -th differential

$$-\alpha_{i+1,j+1} + f_{i+1,j} + \beta_{ij},$$

as desired. \square

Let \mathcal{A} be a DG-category. Recall that we denote by $\{-\}$ the convolution functor $\text{Pre-Tr Mod-}\mathcal{A} \rightarrow \text{Mod-}\mathcal{A}$.

Corollary 2.2. *Let \mathcal{A} be a DG-category and let (E_i, α_{ij}) and (F_i, β_{ij}) be one-sided bounded above twisted complexes over $\text{Mod-}\mathcal{A}$. Let $f = (f_{ij})$ be a one-sided closed map $(E_i, \alpha_{ij}) \rightarrow (F_j, \beta_{ij})$ of degree 0.*

If each component $f_{ii} : E_i \rightarrow F_i$ is a quasi-isomorphism, then so is the induced map

$$\{E_i, \alpha_{ij}\} \xrightarrow{f} \{F_i, \beta_{ij}\}.$$

Proof. By assumption each $\{E_i \xrightarrow{(-1)^i f_{ii}} F_i\}$ is acyclic. Thus the convolution of the twisted complex $\text{Tot}(G_i, \gamma_{ij})$ constructed in Lemma 2.1 is acyclic. This is because it is the convolution of the twisted complex $(\{G_i\}, \gamma_{ij})$ which is a bounded above complex of acyclic modules, cf. the diagram 3.1 in [BK90, §3]. Hence by Lemma 2.1 the convolution of $(\{E_i, \alpha_{ij}\} \xrightarrow{f} \{F_i, \beta_{ij}\})$ is also acyclic. The claim follows. \square

3. PRELIMINARIES ON A_∞ -CATEGORIES

For an introduction to A_∞ -categories we recommend [Kel06], and for a comprehensive technical text – [LH03]. We refer the reader to the latter for the definitions and the notation we employ. Below, we summarise some of it and prove several minor new results.

3.1. A_∞ -algebras, modules, and bimodules. An A_∞ -algebra over k is a graded k -bimodule A together with graded maps

$$m_i : A^{\otimes i} \rightarrow A \quad i \geq 1 \tag{3.1}$$

of degree $2 - i$ which lift to a differential on the augmented tensor coalgebra

$$T^c(A[1]) = \bigoplus_{n \geq 0} A^{\otimes n}[n]$$

generated by $A[1]$. The resulting augmented DG-coalgebra structure on $T^c(A[1])$ is the *bar construction* $B_\infty A$ of A . We denote by τ the counit $B_\infty A \rightarrow k$. The *reduced bar construction* $B_\infty^{\text{red}} A$ is the corresponding DG-coalgebra structure on $T^c(A[1])^{\text{red}} = \bigoplus_{n \geq 1} A^{\otimes n}[n]$.

Let A and B be A_∞ -algebras. An A_∞ -algebra morphism $f : A \rightarrow B$ is a collection of graded maps

$$f_i : A^{\otimes i} \rightarrow B \quad i \geq 1 \tag{3.2}$$

of degree $1 - i$ which lift to a DG-coalgebra morphism $B_\infty A \rightarrow B_\infty B$.

Let A be an A_∞ -algebra. A (right) A -module is a graded k -module E together with a collection of graded maps

$$m_i : E \otimes A^{\otimes i-1} \rightarrow E \quad i \geq 1 \tag{3.3}$$

of degree $2 - i$ which lift to a differential on the free graded $B_\infty A$ -comodule $E[1] \otimes_k B_\infty A$ generated by $E[1]$. The resulting DG $B_\infty A$ -comodule is the *bar construction* $B_\infty E$ of E .

Let E and F be two A -modules. An A_∞ -module morphism $f: E \rightarrow F$ is a collection of graded maps

$$f_i: E \otimes_k A^{\otimes i-1} \rightarrow F \quad i \geq 1 \quad (3.4)$$

of degree $1-i$ which lift to a $B_\infty A$ -comodule morphism $B_\infty E \rightarrow B_\infty F$. An A_∞ -module morphism is *strict* if $f_i = 0$ for $i \geq 2$.

Let A and B be A_∞ -algebras. An A - B -bimodule is a graded k -bimodule M together with a differential $m_{0,0}$ and the action maps

$$m_{i,j}: A^i \otimes_k M \otimes_k B^j \rightarrow M \quad i \geq 0, j \geq 0 \quad (3.5)$$

of degree $1-i-j$ which together lift to a differential

$$(B_\infty A) \otimes_k M[1] \otimes_k (B_\infty B) \longrightarrow (B_\infty A) \otimes_k M[1] \otimes_k (B_\infty B), \quad (3.6)$$

cf. [LH03, §2.5.1]. The resulting DG $B_\infty A$ - $B_\infty B$ -bicomodule is the *bar construction* $B_\infty M$ of M . Whenever it is necessary to avoid confusion, e.g. in the case of the diagonal bimodule, we will denote the bimodule bar construction as $B_\infty^{\text{bim}} M$.

We also consider the partial bar constructions. The B -bar construction $B_\infty^B M$ is the right DG $(B_\infty B)$ -comodule whose underlying graded $B_\infty B$ -comodule is $M \otimes_k B_\infty B$ and whose differential is constructed from $m_{0,j}$ for $j \geq 0$. It also carries the structure of a left $A_\infty A$ -module, defined by $m_{i,j}$ for $i \geq 1, j \geq 0$. Likewise, the A -bar construction $B_\infty^A M$ is the left DG $(B_\infty A)$ -comodule and right $A_\infty B$ -module defined similarly. Consider now DG-algebras $\text{End}_{B_\infty B}(B_\infty^B M)$ and $\text{End}_{(B_\infty A)^{\text{opp}}}(B_\infty^A M)$. Specifying the structure of A - B -bimodule on M is equivalent to specifying the natural A -action A_∞ -morphism

$$A \xrightarrow{\text{act}_M} \text{End}_{B_\infty B}(B_\infty^B M). \quad (3.7)$$

Similarly, it is equivalent to specifying the natural B -action A_∞ -morphism

$$B^{\text{opp}} \xrightarrow{\text{act}_M} \text{End}_{(B_\infty A)^{\text{opp}}}(B_\infty^A M), \quad (3.8)$$

cf. the “lemme clef” of [LH03, §5.3]. Explicitly, we define e.g. the A -action morphism by setting for each $a_1 \otimes \cdots \otimes a_n \in A^n$ the endomorphism $\text{act}_M(a_1 \otimes \cdots \otimes a_n) \in \text{End}_{B_\infty B}(B_\infty^B M)$ to be

$$m \otimes b_1 \otimes \cdots \otimes b_m \mapsto \sum_{l=0}^m (-1)^? m_{n,l}(a_1 \otimes \cdots \otimes a_n \otimes m \otimes b_1 \otimes \cdots \otimes b_l) \otimes b_{l+1} \otimes \cdots \otimes b_m \quad (3.9)$$

where the signs are dictated by the definitions in [LH03, §5.3].

The *diagonal bimodule* A is defined by the graded maps

$$m_{i,j}: A^i \otimes_k A \otimes_k A^j \xrightarrow{m_{i+j+1}} A \quad i \geq 0, j \geq 0 \quad (3.10)$$

where m_i are the maps which define A_∞ -algebra structure on A .

Let M and N be two A - B -bimodules. An A_∞ -bimodule morphism $f: M \rightarrow N$ is a collection of graded maps

$$f_{i,j}: A^{\otimes i} \otimes_k M \otimes_k B^{\otimes j} \rightarrow N \quad i \geq 0, j \geq 0 \quad (3.11)$$

of degree $-i-j$ which lift to a $B_\infty A$ - $B_\infty B$ -bicomodule morphism $B_\infty M \rightarrow B_\infty N$. An A_∞ -module morphism is *strict* if $f_{i,j} = 0$ for $i \geq 1$ or $j \geq 1$.

The DG k -module $\text{Hom}_{B_\infty B}(B_\infty^B M, B_\infty^B N)$ has a natural structure of an A - A -bimodule defined via A -action maps for $B_\infty^B M$ and $B_\infty^B N$. Similar to (3.7) and (3.8), specifying an A_∞ A - B -bimodule morphism $M \xrightarrow{f} N$ is then equivalent to specifying a DG $B_\infty A$ - $B_\infty A$ -bicomodule morphism

$$B_\infty A \xrightarrow{f_A} B_\infty \left(\text{Hom}_{B_\infty B}(B_\infty^B M, B_\infty^B N) \right). \quad (3.12)$$

Similarly, it is equivalent to specifying a DG $B_\infty B$ - $B_\infty B$ -bicomodule morphism

$$B_\infty B \xrightarrow{f_B} B_\infty \left(\text{Hom}_{B_\infty A}(B_\infty^A M, B_\infty^A N) \right). \quad (3.13)$$

3.2. A_∞ -categories. Let \mathbb{A} be a set. We define $k_{\mathbb{A}}$ to be the category whose set of objects is \mathbb{A} and whose morphisms spaces are

$$\mathrm{Hom}_{k_{\mathbb{A}}}(a, b) = \begin{cases} k & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases} \quad (3.14)$$

For any sets \mathbb{A}, \mathbb{B} and \mathbb{C} , any graded $k_{\mathbb{A}}\text{-}k_{\mathbb{B}}$ -bimodule M , and $k_{\mathbb{B}}\text{-}k_{\mathbb{C}}$ -bimodule N we denote by \otimes_k their tensor product over $k_{\mathbb{B}}$:

$${}_a(M \otimes_k N)_c = \bigoplus_{b \in \mathbb{B}} {}_a M_b \otimes_k {}_b M_c. \quad (3.15)$$

We give the category of graded $k_{\mathbb{A}}\text{-}k_{\mathbb{A}}$ -bimodules a monoidal structure by equipping it with the multiplication given by \otimes_k and the identity element given by the diagonal bimodule $k_{\mathbb{A}}$.

Given a graded $k_{\mathbb{A}}\text{-}k_{\mathbb{B}}$ -bimodule M and two maps of sets $\mathbb{A}' \xrightarrow{f} \mathbb{A}$ and $\mathbb{B}' \xrightarrow{g} \mathbb{B}$, we write ${}_f M_g$ for the graded $k_{\mathbb{A}'}\text{-}k_{\mathbb{B}'}$ -bimodule obtained by pulling back along f and g , i.e.

$${}_{a'}({}_f M_g)_{b'} = {}_{f(a')} M_{g(b')} \quad \forall a' \in \mathbb{A}', b' \in \mathbb{B}'. \quad (3.16)$$

An A_∞ -category \mathcal{A} is an object set \mathbb{A} and an A_∞ -algebra \mathcal{A} over $k_{\mathbb{A}}$, i.e. in the category of graded $k_{\mathbb{A}}$ -bimodules. We abuse the notation by also using \mathcal{A} to denote the object set \mathbb{A} where it doesn't cause confusion, e.g. we write $k_{\mathcal{A}}$ for $k_{\mathbb{A}}$.

Given two A_∞ -categories \mathcal{A} and \mathcal{B} an A_∞ -functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a map $\mathbb{A} \xrightarrow{F} \mathbb{B}$ of their object sets and a morphism $\mathcal{A} \rightarrow {}_F \mathcal{B}_F$ of A_∞ -algebras in the category of graded $k_{\mathcal{A}}\text{-}k_{\mathcal{A}}$ -bimodules.

The definitions of modules, bimodules, etc. for A_∞ -algebras in §3.1 generalise similarly to A_∞ -categories by considering the latter as A_∞ -algebras in appropriate categories of graded bimodules, cf. [LH03, §5.1].

Let \mathcal{A} be A_∞ -category. We denote by $\mathbf{Nod}_\infty \mathcal{A}$ the DG-category of all A_∞ -modules over \mathcal{A} . Its objects are \mathcal{A} -modules and for any two \mathcal{A} -modules E and F we have

$$\mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{A}}(E, F) \simeq \mathrm{Hom}_{B_\infty \mathcal{A}}(B_\infty E, B_\infty F), \quad (3.17)$$

cf. [LH03, §5.2]. It follows that the elements of $\mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{A}}(E, F)$ can be identified with arbitrary collections of graded $k_{\mathcal{A}}$ -module morphisms $\{E \otimes_k \mathcal{A}^{\otimes i} \rightarrow F\}_{i \geq 0}$. Note that such collection defines a morphism of A_∞ -modules, as per §3.1, if and only if the corresponding element is closed of degree 0.

Let \mathcal{A} and \mathcal{B} be A_∞ -categories. The DG-category $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$ of A_∞ $\mathcal{A}\text{-}\mathcal{B}$ -bimodules is defined similarly to the above.

Let M be an $\mathcal{A}\text{-}\mathcal{B}$ bimodule. The notions of partial bar constructions and action maps defined in §3.1 extend to A_∞ -functors $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ and $\mathcal{B}^{\mathrm{opp}} \rightarrow \mathbf{Nod}_\infty \mathcal{A}^{\mathrm{opp}}$, cf. [LH03, Cor. 5.3.0.2]. Given $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we write ${}_a M$ and M_b for their images under these functors. When \mathcal{A} and \mathcal{B} are A_∞ -algebras, i.e. A_∞ -categories with a single object \bullet , ${}_a M$ is e.g. the \mathcal{B} -module which corresponds to $B_\infty^{\mathcal{B}} M$, and the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ acts on the morphism spaces by the A_∞ -morphism $\mathcal{A} \rightarrow \mathrm{Hom}_{\mathbf{Nod}_\infty \mathcal{B}}(\bullet M, \bullet M)$ which corresponds to the action map $\mathcal{A} \xrightarrow{\mathrm{act}_M} \mathrm{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M)$.

In case when M is the diagonal bimodule \mathcal{A} , this yields the *Yoneda embedding* $\mathcal{A} \hookrightarrow \mathbf{Nod}_\infty \mathcal{A}$. The modules $\{{}_a \mathcal{A}\}_{a \in \mathcal{A}}$ are the *representable* \mathcal{A} -modules. Explicitly, the graded $k_{\mathcal{A}}$ -module underlying ${}_a \mathcal{A}$ is $\mathrm{Hom}_{\mathcal{A}}(-, a)$ and its A_∞ -module structure is given by the A_∞ -operations m_i of \mathcal{A} .

An \mathcal{A} -module E is *free* if it is isomorphic to a direct sum of shifts of representable modules. An \mathcal{A} -module E is *semifree* if it admits an ascending filtration whose quotients are free modules.

Definition 3.1. Let \mathcal{A} be an A_∞ -category and N be a DG $k_{\mathcal{A}}$ -module. We denote by $N \otimes_k \mathcal{A}$ the \mathcal{A} -module generated by N . It is the A_∞ \mathcal{A} -module which corresponds to the free graded $B_\infty \mathcal{A}$ -comodule

$$N \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}$$

equipped with the differential $d_N \otimes d_{B_\infty^{\mathrm{red}} \mathcal{A}}$. Explicitly, it has $N \otimes_k \mathcal{A}$ as the underlying DG k -module and for each $n \geq 2$ we have

$$m_{n+1}^{N \otimes_k \mathcal{A}}(n \otimes a, a_1, \dots, a_n) = (-1)^n n \otimes m_{n+1}(a, a_1, \dots, a_n) \quad n \otimes a \in N \otimes_k \mathcal{A}, a_i \in \mathcal{A}$$

with appropriate signs.

For any $E \in \mathbf{Nod}_\infty \mathcal{A}$ let $E \otimes_k \mathcal{A}$ be the above construction applied to the DG $k_{\mathcal{A}}$ -module underlying E .

Lemma 3.2. Let \mathcal{A} be an A_∞ -category and let N be a DG $k_{\mathcal{A}}$ -module. The \mathcal{A} -module $N \otimes_k \mathcal{A}$ is semifree.

Proof. Suppose first that N is a graded $k_{\mathcal{A}}$ -module considered as a DG-module with zero differential. Then $N \otimes_k \mathcal{A}$ is a free \mathcal{A} -module, as it is isomorphic to

$$\bigoplus_{a \in \mathcal{A}, i \in \mathbb{Z}} (N_a)_i \otimes_k a \mathcal{A}[-i].$$

On the other hand, if N is a DG-module bounded from above $N \otimes_k \mathcal{A}$ is clearly semifree as it admits a filtration whose quotients are $N_i \otimes_k \mathcal{A}$. Finally, since k is a field, we can (non-canonically) decompose an arbitrary DG $k_{\mathcal{A}}$ -module N as a direct sum of its graded cohomology module $H^*(N)$ and acyclic DG-modules $\text{Im } d_i \rightarrow \text{Im } d_i$ concentrated in degrees i and $i + 1$. It follows that $N \otimes_k \mathcal{A}$ is semifree, as desired. \square

3.3. The derived category of an A_{∞} -category. Unlike the case of DG-categories, a morphism of A_{∞} -modules is a quasi-isomorphism if and only if it is a homotopy equivalence, cf. [LH03, Prop. 2.4.1.1]. Thus $H^0(\mathbf{Nod}_{\infty} \mathcal{A})$ can be identified with the localisation of the category of A_{∞} -modules and A_{∞} -morphisms by quasi-isomorphisms. However, also unlike the DG case, $H^0(\mathbf{Nod}_{\infty} \mathcal{A})$ is not apriori generated as a triangulated category by representable \mathcal{A} -modules. Thus the *derived category* $D(\mathcal{A})$ of \mathcal{A} is defined to be the smallest full subcategory of $H^0(\mathbf{Nod}_{\infty} \mathcal{A})$ which is triangulated, cocomplete, closed under isomorphisms, and contains the representable modules. Note that if \mathcal{A} is a DG-category, this agrees with the usual definition of the derived category of a DG-category. We denote by $(\mathbf{Nod}_{\infty} \mathcal{A})_{hu}$ the full subcategory of $\mathbf{Nod}_{\infty} \mathcal{A}$ consisting of the modules whose images in $H^0(\mathbf{Nod}_{\infty} \mathcal{A})$ lie in $D(\mathcal{A})$.

For any $E \in \mathbf{Nod}_{\infty} \mathcal{A}$ the conditions are equivalent:

- (1) E is homotopic to a semi-free module.
- (2) E lies in $(\mathbf{Nod}_{\infty} \mathcal{A})_{hu}$.
- (3) E is H -unitary, that is — its bar-construction $B_{\infty} M$ is acyclic.

The equivalence of (1) and (2) is straightforward, while that of (2) and (3) is due to [LH03, Prop. 4.1.2.10].

Let now \mathcal{A} be a strictly unital A_{∞} -category. By [LH03, Prop. 4.1.3.7] the equivalent conditions above are further equivalent to:

- (4) E is *homologically unitary*, that is — $H^*(E)$ is a unitary graded $H^*(\mathcal{A})$ -module.

We thus have a chain of inclusions

$$\mathbf{Mod}_{\infty} \mathcal{A} \hookrightarrow (\mathbf{Nod}_{\infty} \mathcal{A})_u \hookrightarrow (\mathbf{Nod}_{\infty} \mathcal{A})_{hu} \hookrightarrow \mathbf{Nod}_{\infty} \mathcal{A}$$

where $(\mathbf{Nod}_{\infty} \mathcal{A})_u$ is the full subcategory consisting of strictly unital modules, and $\mathbf{Mod}_{\infty} \mathcal{A}$ is its non-full subcategory of strictly unital modules and strictly unital morphisms between them. The first two inclusions are quasi-equivalences, and thus $\mathbf{Mod}_{\infty} \mathcal{A}$ and $(\mathbf{Nod}_{\infty} \mathcal{A})_u$ are alternative DG-enhancements of $D_{\infty}(\mathcal{A})$.

The derived categories of A_{∞} -bimodules are defined similarly and similar considerations apply.

3.4. Tensor and Hom functors for bimodules. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be A_{∞} -categories. Let $M \in \mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B}$, and $N \in \mathbf{Nod}_{\infty} \mathcal{B}\text{-}\mathcal{C}$. We define the A_{∞} -tensor product $M \overset{\infty}{\otimes} N$ to be the A_{∞} $\mathcal{A}\text{-}\mathcal{C}$ -bimodule whose bar construction is the (shifted) cotensor product of DG-comodules

$$B_{\infty} M \otimes_{B_{\infty} \mathcal{B}} B_{\infty} N[-1]. \quad (3.18)$$

Explicitly, the underlying graded $k_{\mathcal{A}}\text{-}k_{\mathcal{C}}$ -bimodule is

$$M \otimes_k (B_{\infty} \mathcal{B}) \otimes_k N \quad (3.19)$$

and its A_{∞} $\mathcal{A}\text{-}\mathcal{C}$ -bimodule structure consists of the differential

$$m_{0,0} = -d_{B_{\infty} M} \otimes \text{Id} - \text{Id} \otimes d_{B_{\infty} N} + \text{Id} \otimes d_{B_{\infty} \mathcal{B}} \otimes \text{Id}$$

and of commuting \mathcal{A} and \mathcal{C} actions induced from those on M and N respectively:

$$m_{l,n}((a_1 \otimes \cdots \otimes a_l) \otimes m \otimes (b_1 \otimes \cdots \otimes b_m) \otimes n \otimes (c_1 \otimes \cdots \otimes c_n))$$

equals

$$\begin{cases} 0 & \text{if } l, n \neq 0 \\ \bigoplus_{i=0}^m (-1)^? m_{l,i}^E(a_1 \otimes \cdots \otimes a_l \otimes m \otimes b_1 \otimes \cdots \otimes b_i) \otimes (b_{i+1} \otimes \cdots \otimes b_m) \otimes n & \text{if } l \neq 0, n = 0 \\ \bigoplus_{i=0}^m (-1)^? m \otimes (b_1 \otimes \cdots \otimes b_i) \otimes m_{m-i,n}^F(b_{i+1} \otimes \cdots \otimes b_m \otimes n \otimes c_1 \otimes \cdots \otimes c_n) & \text{if } l = 0, n \neq 0 \end{cases} \quad (3.20)$$

with the signs dictated by (3.18).

Let now $M \xrightarrow{f} M'$ be a morphism in $\mathbf{Nod}_{\infty} \mathcal{A}\text{-}\mathcal{B}$ and $N \xrightarrow{g} N'$ to be a morphism in $\mathbf{Nod}_{\infty} \mathcal{B}\text{-}\mathcal{C}$. Define the morphism

$$M \overset{\infty}{\otimes}_{\mathcal{B}} N \xrightarrow{f \otimes g} M' \overset{\infty}{\otimes}_{\mathcal{B}} N'$$

to be the morphism in $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{C}$ which corresponds to the DG-bicomodule morphism

$$B_\infty M \otimes_{B_\infty \mathcal{B}} B_\infty N \xrightarrow{f \otimes g} B_\infty M' \otimes_{B_\infty \mathcal{B}} B_\infty N'.$$

Explicitly, $f \otimes g$ sends

$$(a_1 \otimes \cdots \otimes a_l) \otimes m \otimes (b_1 \otimes \cdots \otimes b_m) \otimes n \otimes (c_1 \otimes \cdots \otimes c_n)$$

to

$$\bigoplus_{0 \leq i \leq j \leq m} (-1)^? f_{i,i} (a_1 \otimes \cdots \otimes a_l \otimes m \otimes b_1 \otimes \cdots \otimes b_i) \otimes (b_{i+1} \otimes \cdots \otimes b_j) \otimes g_{m-j,n} (b_{j+1} \otimes \cdots \otimes b_m \otimes n \otimes c_1 \otimes \cdots \otimes c_n).$$

We thus obtain a DG-functor:

$$(-)_{\otimes \mathcal{B}}^\infty (-): \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B} \otimes_k \mathbf{Nod}_\infty \mathcal{B}\text{-}\mathcal{C} \longrightarrow \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{C}. \quad (3.21)$$

Note that $f \otimes \text{Id}$ is \mathcal{C} -strict: $(f \otimes \text{Id})_{i,j} = 0$ if $j > 0$. It follows that for any $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$ the functor

$$(-)_{\otimes \mathcal{A}}^\infty M: \mathbf{Nod}_\infty \mathcal{A} \longrightarrow \mathbf{Nod}_\infty \mathcal{B}$$

filters through the non-full subcategory $\mathbf{Nod}_\infty^{\text{strict}} \mathcal{B} \subset \mathbf{Nod}_\infty \mathcal{B}$ consisting of all \mathcal{B} -modules and strict A_∞ -morphisms between them.

Now let $L \in \mathbf{Nod}_\infty \mathcal{D}\text{-}\mathcal{B}$, and $M \in \mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$. We define the A_∞ -Hom bimodule $\text{Hom}_\mathcal{B}^\infty(L, M)$ as follows. The underlying graded $k_{\mathcal{A}}\text{-}k_{\mathcal{D}}$ -bimodule is

$$\text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L, B_\infty^\mathcal{B} M). \quad (3.22)$$

It has a natural structure of a DG-bimodule over DG-categories $\text{End}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} M)$ and $\text{End}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L)$. Using the \mathcal{A} - and \mathcal{D} -action functors we restrict this to an A_∞ $\mathcal{A}\text{-}\mathcal{D}$ -bimodule structure, cf. [Kel01, §6.2].

Explicitly, this bimodule structure consists of the standard differential

$$m_{0,0}(\alpha) = d_{B_\infty^\mathcal{B} F} \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_{B_\infty^\mathcal{B} E} \quad (3.23)$$

and of commuting \mathcal{A} and \mathcal{C} actions: for any $\alpha \in \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} E, B_\infty^\mathcal{B} F)$ we have

$$m_{l,n}((a_1 \otimes \cdots \otimes a_l) \otimes \alpha \otimes (c_1 \otimes \cdots \otimes c_n)) = \begin{cases} 0 & \text{if } l, n \neq 0 \\ (-1)^? \text{act}_M(a_1 \otimes \cdots \otimes a_l) \circ \alpha & \text{if } l \neq 0, n = 0 \\ (-1)^? \alpha \circ \text{act}_M(c_1 \otimes \cdots \otimes c_n) & \text{if } l = 0, n \neq 0. \end{cases}$$

where the signs are dictated by the definition of the restriction functor in [Kel01, §6.2].

Let now further $N \in \mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{B}$. It can be readily checked that the *composition map*

$$B_\infty(\text{Hom}_\mathcal{B}^\infty(M, N)) \otimes_{B_\infty \mathcal{A}} B_\infty(\text{Hom}_\mathcal{B}^\infty(L, M)) \xrightarrow{\text{cmps}} B_\infty(\text{Hom}_{B_\infty \mathcal{B}}^\infty(L, N)) \quad (3.24)$$

defined by

$$\begin{aligned} & B_\infty \mathcal{C} \otimes_k \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} M, B_\infty^\mathcal{B} N) \otimes_k B_\infty \mathcal{A} \otimes_k \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L, B_\infty^\mathcal{B} M) \otimes_k B_\infty \mathcal{D} \xrightarrow{\text{Id}^{\otimes 2} \otimes \tau \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow B_\infty \mathcal{C} \otimes_k \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} M, B_\infty^\mathcal{B} N) \otimes_k \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L, B_\infty^\mathcal{B} M) \otimes_k B_\infty \mathcal{D} \xrightarrow{\text{Id} \otimes \text{cmps} \otimes \text{Id}} \\ & \rightarrow B_\infty \mathcal{C} \otimes_k \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L, B_\infty^\mathcal{B} N) \otimes_k B_\infty \mathcal{D} \end{aligned}$$

commutes with the differentials. It defines therefore in $\mathbf{Nod}_\infty \mathcal{C}\text{-}\mathcal{D}$ the *composition map*

$$\text{Hom}_\mathcal{B}^\infty(M, N)_{\otimes \mathcal{A}}^\infty \text{Hom}_\mathcal{B}^\infty(L, M) \xrightarrow{\text{cmps}} \text{Hom}_\mathcal{B}^\infty(L, N). \quad (3.25)$$

Let now $L' \xrightarrow{f} L$ and $M \xrightarrow{g} M'$ be morphisms in $\mathbf{Nod}_\infty \mathcal{D}\text{-}\mathcal{B}$ and $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{B}$, respectively. Define the morphism

$$\text{Hom}_\mathcal{B}^\infty(L, M) \xrightarrow{g \circ (-) \circ f} \text{Hom}_\mathcal{B}^\infty(L', M')$$

in $\mathbf{Nod}_\infty \mathcal{A}\text{-}\mathcal{C}$ by the DG bicomodule morphism

$$\begin{aligned} & B_\infty(\text{Hom}_\mathcal{B}^\infty(L, M)) \simeq B_\infty \mathcal{A} \otimes_{B_\infty \mathcal{A}} B_\infty(\text{Hom}_\mathcal{B}^\infty(L, M)) \otimes_{B_\infty \mathcal{D}} B_\infty \mathcal{D} \xrightarrow{g_{\mathcal{A}} \otimes \text{Id} \otimes f_{\mathcal{D}}} \\ & \rightarrow B_\infty(\text{Hom}_\mathcal{B}^\infty(M, M')) \otimes_{B_\infty \mathcal{A}} B_\infty(\text{Hom}_\mathcal{B}^\infty(L, M)) \otimes_{B_\infty \mathcal{D}} B_\infty(\text{Hom}_\mathcal{B}^\infty(L', L)) \xrightarrow{\text{cmps}} B_\infty(\text{Hom}_\mathcal{B}^\infty(L', M')). \end{aligned}$$

Explicitly, for any $\alpha \in \text{Hom}_{B_\infty \mathcal{B}}(B_\infty^\mathcal{B} L, B_\infty^\mathcal{B} M)$ the map $(f \circ - \circ g)_{l,n}$ sends

$$(a_1 \otimes \cdots \otimes a_l) \otimes \alpha \otimes (d_1 \otimes \cdots \otimes d_n)$$

to the map

$$(-1)^? (g_{l,\bullet} (a_1 \otimes \cdots \otimes a_l \otimes -) \otimes \text{Id}) \circ \Delta \circ \alpha \circ (f_{l,\bullet} (a_1 \otimes \cdots \otimes a_l \otimes -) \otimes \text{Id}) \circ \Delta$$

in $\text{Hom}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} L', B_\infty^{\mathcal{B}} M')$. Here Δ denotes, as usual, the comodule comultiplications.

We thus obtain a DG-functor

$$\text{Hom}_{\mathcal{B}}^\infty(-, -): (\mathbf{Nod}_\infty \mathcal{C} - \mathcal{B})^{\text{opp}} \otimes \mathbf{Nod}_\infty \mathcal{A} - \mathcal{B} \rightarrow \mathbf{Nod}_\infty \mathcal{A} - \mathcal{C} \quad (3.26)$$

and, similar to the above, for any $M \in \mathbf{Nod}_\infty \mathcal{A} - \mathcal{B}$ the functor

$$\text{Hom}_{\mathcal{B}}^\infty(M, -): \mathbf{Nod}_\infty \mathcal{B} \longrightarrow \mathbf{Nod}_\infty \mathcal{A}$$

filters through $\mathbf{Nod}_\infty^{\text{strict}} \mathcal{A} \subset \mathbf{Nod}_\infty \mathcal{A}$.

We then have the usual Tensor-Hom adjunction: for every $M \in \mathbf{Nod}_\infty \mathcal{A} - \mathcal{B}$ the functors

$$(-)_{\otimes \mathcal{A}}^\infty M: \mathbf{Nod}_\infty \mathcal{C} - \mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{C} - \mathcal{B} \quad (3.27)$$

$$\text{Hom}_{\mathcal{B}}^\infty(M, -): \mathbf{Nod}_\infty \mathcal{C} - \mathcal{B} \rightarrow \mathbf{Nod}_\infty \mathcal{C} - \mathcal{A} \quad (3.28)$$

are left and right adjoint to each other, respectively. Same holds for the functors $M_{\otimes \mathcal{B}}^\infty(-)$ and $\text{Hom}_{\mathcal{A}^{\text{opp}}}^\infty(M, -)$.

Let $M \in \mathbf{Nod}_\infty \mathcal{A} - \mathcal{B}$. The DG $k_{\mathcal{A}} - k_{\mathcal{A}}$ -bimodule underlying $\text{Hom}_{\mathcal{B}}^\infty(M, M)$ has an algebra structure given by composition, and thus defines a DG-category with the same object set as \mathcal{A} . By definition, this DG-category can be naturally identified with the DG-category $\text{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M)$. On the other hand, it can be identified with the image of the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ defined by M . Indeed, the assignment $a \mapsto {}_a M$ gives a fully faithful inclusion $\text{Hom}_{\mathcal{B}}^\infty(M, M) \hookrightarrow \mathbf{Nod}_\infty \mathcal{B}$, and the functor $\mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{B}$ decomposes as

$$\mathcal{A} \xrightarrow{\text{act}_M} \text{Hom}_{\mathcal{B}}^\infty(M, M) \hookrightarrow \mathbf{Nod}_\infty \mathcal{B}. \quad (3.29)$$

Here we write act_M for the composition $\mathcal{A} \xrightarrow{\text{act}_M} \text{End}_{B_\infty \mathcal{B}}(B_\infty^{\mathcal{B}} M) \simeq \text{Hom}_{\mathcal{B}}^\infty(M, M)$.

3.5. A functorial semifree resolution for $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$. To our knowledge, the material presented in this section is original to this paper.

Let \mathcal{A} be an A_∞ -category and let $E \in \mathbf{Nod}_\infty \mathcal{A}$. Consider the A_∞ \mathcal{A} -module $E_{\otimes \mathcal{A}}^\infty \mathcal{A}$. The corresponding DG $B_\infty \mathcal{A}$ -comodule is

$$B_\infty E \otimes_{B_\infty \mathcal{A}} B_\infty^{\text{bim}} \mathcal{A} [-1]. \quad (3.30)$$

As a graded $B_\infty \mathcal{A}$ -comodule (3.30) is isomorphic to

$$E \otimes_k B_\infty \mathcal{A} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}$$

which decomposes as

$$\bigoplus_{i \geq 0} E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}.$$

It follows that the differential of (3.30) decomposes into:

- (1) For each $i \geq 0$ a degree 1 square zero $B_\infty \mathcal{A}$ -coderivation

$$E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \longrightarrow E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \quad (3.31)$$

- (2) For each $i > j \geq 0$ a degree 1 graded $B_\infty \mathcal{A}$ -comodule morphism

$$E \otimes_k (\mathcal{A}[1])^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A} \longrightarrow E \otimes_k (\mathcal{A}[1])^{\otimes j} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}. \quad (3.32)$$

For any $i > 0$ write $E \otimes_k \mathcal{A}^{\otimes i+1}$ for the A_∞ \mathcal{A} -module

$$(E \otimes_k \mathcal{A}^{\otimes i}) \otimes_k \mathcal{A}$$

in the sense of Definition 3.1. The corresponding DG $B_\infty \mathcal{A}$ -comodule is the graded $B_\infty \mathcal{A}$ -comodule

$$E \otimes_k \mathcal{A}^{\otimes i} \otimes_k \mathcal{A}[1] \otimes_k B_\infty \mathcal{A}$$

whose differential is (the shift of) the coderivation (3.31).

Definition 3.3. For any $i > 0$ define a degree $i + 1$ morphism

$$E \otimes_k \mathcal{A}^{\otimes i} \longrightarrow E \quad (3.33)$$

in $\mathbf{Nod}_\infty \mathcal{A}$ by the graded $k_{\mathcal{A}}$ -module maps

$$E \otimes_k \mathcal{A}^{\otimes i} \otimes_k \mathcal{A}^{\otimes n} \xrightarrow{m_{n+i+1}} E.$$

For any $i > j > 0$ define a degree $i - j + 1$ morphism

$$E \otimes_k \mathcal{A}^{\otimes i} \longrightarrow E \otimes_k \mathcal{A}^{\otimes j} \quad (3.34)$$

in $\mathbf{Nod}_\infty \mathcal{A}$ by the (shift of the) graded $B_\infty \mathcal{A}$ -comodule morphism (3.32).

Explicitly, (3.34) is defined by the maps

$$f_{n+1}: E \otimes_k \mathcal{A}^{\otimes(i-1)} \otimes_k \mathcal{A} \otimes_k \mathcal{A}^{\otimes n} \longrightarrow E \otimes_k \mathcal{A}^{\otimes(j-1)} \otimes_k \mathcal{A}$$

where

$$f_1 = \sum_{\substack{r+1+s=j+1 \\ r \geq 0, s \geq 0}} -1^? \text{Id}^{\otimes r} \otimes m_{i+1-r-s} \otimes \text{Id}^{\otimes s}.$$

and for any $n \geq 1$

$$f_{n+1} = -1^? \text{Id}^{\otimes j} \otimes m_{i-j+1+n}.$$

Lemma 3.4. Let \mathcal{A} be an A_∞ -category. The functor

$$(-)^{\otimes_{\mathcal{A}}} \mathcal{A}: \mathbf{Nod}_\infty \mathcal{A} \rightarrow \mathbf{Nod}_\infty \mathcal{A}$$

filters through the full subcategory $\mathcal{SF}^{\text{strict}}(\mathcal{A}) \subset \mathbf{Nod}_\infty \mathcal{A}$ consisting of semifree modules and strict A_∞ -morphisms between them.

Proof. As explained in §3.4 for any $M \in \mathbf{Nod}_\infty \mathcal{A} \mathcal{B}$ the functor

$$(-)^{\otimes_{\mathcal{A}}} M: \mathbf{Nod}_\infty \mathcal{A} \longrightarrow \mathbf{Nod}_\infty \mathcal{B}$$

filters through $\mathbf{Nod}_\infty^{\text{strict}} \mathcal{B}$. It remains to show that for any $E \in \mathbf{Nod}_\infty \mathcal{A}$ the module $E^{\otimes_{\mathcal{A}}} \mathcal{A}$ is semifree.

Recall the decomposition of the differential on the DG comodule corresponding to $E^{\otimes_{\mathcal{A}}} \mathcal{A}$ discussed prior to and employed in Definition 3.3. It follows tautologically that $E^{\otimes_{\mathcal{A}}} \mathcal{A}$ is isomorphic to the convolution of the bounded above, one-sided twisted complex

$$\begin{array}{c} \cdots \longrightarrow E \otimes_k \mathcal{A}^{\otimes 3} \longrightarrow E \otimes_k \mathcal{A}^{\otimes 2} \longrightarrow E \otimes_k \mathcal{A} \\ \text{deg. 0} \end{array} \quad (3.35)$$

whose differentials are the A_∞ -morphisms (3.34).

Since k is a field, each of the modules $E \otimes_k \mathcal{A}^{\otimes i}$ in the twisted complex (3.35) is semifree by Lemma 3.2. We conclude that $E^{\otimes_{\mathcal{A}}} \mathcal{A}$ is semifree, as it is isomorphic to the convolution of a bounded from above, one-sided twisted complex of semifree modules. \square

Corollary 3.5. Let \mathcal{A} be a strictly unital A_∞ -category (resp. a DG-category). The functor

$$(-)^{\otimes_{\mathcal{A}}} \mathcal{A}: \mathbf{Nod}_\infty \mathcal{A} \rightarrow \mathbf{Nod}_\infty^{\text{strict}} \mathcal{A}$$

filters through the full subcategory of $\mathbf{Nod}_\infty^{\text{strict}} \mathcal{A}$ consisting of strictly unital (resp. DG) modules.

NB: When \mathcal{A} is a DG-category, strict A_∞ -morphisms between DG-modules are simply the DG-morphisms, so the subcategory of $\mathbf{Nod}_\infty^{\text{strict}} \mathcal{A}$ consisting of DG-modules is canonically isomorphic to the usual DG-category $\mathbf{Mod}\text{-}\mathcal{A}$ of DG-modules over \mathcal{A} .

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\quad} & E \otimes_k \mathcal{A}^{\otimes 3} & \xrightarrow{\quad} & E \otimes_k \mathcal{A}^{\otimes 2} & \xrightarrow{\quad} & E \otimes_k \mathcal{A} \\
& & & & & & \text{deg.0} \\
& & & & & & \downarrow \\
& & & & & & E \\
& & & & & & \text{deg.0}
\end{array}
\tag{3.36}$$
$$(-) \overset{\infty}{\otimes}_{\mathcal{A}} \mathcal{A} \longrightarrow \text{Id} . \quad (3.37)$$

Let \mathcal{A} be a DG-category and let $\mathbf{Mod}\text{-}\mathcal{A}$ be the DG-category of \mathcal{A} -modules. There are two enhancements commonly used in the literature for $D(\mathcal{A})$: the full subcategory $\mathcal{P}(\mathcal{A})$ of the h -projective modules in $\mathbf{Mod}\text{-}\mathcal{A}$, and the Drinfeld quotient $\mathbf{Mod}\text{-}\mathcal{A}/\mathcal{Ac}(\mathcal{A})$ where $\mathcal{Ac}(\mathcal{A})$ is the full subcategory of acyclic modules. Neither turned out to be suitable for our purposes. The problem with the Drinfeld quotient is that its morphisms are inconvenient to work with explicitly. The problem with $\mathcal{P}(\mathcal{A})$ manifests itself when working with bimodules. The diagonal bimodule \mathcal{A} , which corresponds to the identity functor $D(\mathcal{A}) \rightarrow D(\mathcal{A})$, is not h -projective. Hence every construction involving the identity functor has to be h -projectively resolved by e.g. tensoring with the bar complex. This leads to many formulas becoming vastly more complicated than they should be, cf. [AL13].

We propose a different DG-enhancement framework for the derived categories of DG-categories. We think it more suitable for identifying the derived categories of DG-bimodules with triangulated categories of DG-enhanceable functors as described above. Let \mathcal{A} be a DG-category. The proposed enhancement of $D(\mathcal{A})$ admits two different descriptions.

4.1. DG-modules with A_∞ -morphisms between them. The first one is in the language of A_∞ -categories and modules. The enhancement we want is the full subcategory of the DG-category $\mathbf{Nod}_\infty \mathcal{A}$ of A_∞ \mathcal{A} -modules which consists of DG \mathcal{A} -modules. We denote this subcategory by $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$. Note that the subcategory $(\mathbf{Nod}_\infty^{\text{strict}} \mathcal{A})_{dg} \subset \mathbf{Nod}_\infty \mathcal{A}$ which consists of DG \mathcal{A} -modules and strict A_∞ -morphisms between them can be canonically identified with the usual DG-category $\mathbf{Mod}\text{-}\mathcal{A}$ of DG \mathcal{A} -modules. Consider the chain of subcategory inclusions

$$\mathcal{SF}(\mathcal{A}) \hookrightarrow \mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{dg} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{hu}. \quad (4.1)$$

In $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ all quasi-isomorphisms are homotopy equivalences, and thus the functorial resolution $(-)^{\infty}_{\mathcal{A}} \mathcal{A}$ of $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ into $\mathcal{SF}(\mathcal{A})$ established in Cor. 3.8 ensures that every full subcategory of $(\mathbf{Nod}_\infty \mathcal{A})_{hu}$ which contains $\mathcal{SF}(\mathcal{A})$ is quasi-equivalent to it. It follows that $D(\mathcal{A}) \simeq D_\infty(\mathcal{A})$ and that $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ is an alternative DG-enhancement of $D(\mathcal{A})$.

For any DG-bimodule $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ the adjoint functors $(-)^{\infty}_{\mathcal{A}} M$ and $\text{Hom}_{\mathcal{B}}(M, -)^{\infty}$ restrict from $\mathbf{Nod}_\infty \mathcal{A} \leftrightarrow \mathbf{Nod}_\infty \mathcal{B}$ to $(\mathbf{Nod}_\infty \mathcal{A})_{dg} \leftrightarrow (\mathbf{Nod}_\infty \mathcal{B})_{dg}$. We thus have the usual Tensor-Hom adjunction for the categories $(\mathbf{Nod}_\infty)_{dg}$.

4.2. The category $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. The second description is a direct one in the language of DG-modules. While less conceptual, it significantly simplifies the computations involved and allows one to avoid having to deal with the sign conventions for A_∞ -categories and modules.

Definition 4.1. Let \mathcal{A} be a DG-category. Define the *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ as follows:

- The object set of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is the same as that of $\mathbf{Mod}\text{-}\mathcal{A}$: DG-modules over \mathcal{A} .
- For any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ set

$$\text{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F) = \text{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \bar{\mathcal{A}}, F) \quad (4.2)$$

and write $\overline{\text{Hom}}_{\mathcal{A}}(E, F)$ to denote this Hom-complex.

- For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ set $\text{Id}_E \in \overline{\text{Hom}}_{\mathcal{A}}(E, E)$ to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\sim} E \quad (4.3)$$

where $\bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}$ is the canonical projection.

- For any $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ define the composition map

$$\overline{\text{Hom}}_{\mathcal{A}}(F, G) \otimes_k \overline{\text{Hom}}_{\mathcal{A}}(E, F) \longrightarrow \overline{\text{Hom}}_{\mathcal{A}}(E, G) \quad (4.4)$$

by setting for any $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha} F$ and $F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G$ the composition of the corresponding elements to be the element given by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \text{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta} G. \quad (4.5)$$

Let \mathcal{A} and \mathcal{B} be DG-categories. We define the bimodule category $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ similarly, but with

$$\overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, N) = \text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \quad \forall M, N \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}. \quad (4.6)$$

This is compatible with the standard identification of right and left \mathcal{A} -modules with $k\text{-}\mathcal{A}$ - and $\mathcal{A}\text{-}k$ -bimodules.

Proposition 4.2. Let \mathcal{A} be a DG-category. We have a (non-full) inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow \overline{\mathbf{Mod}}\text{-}\mathcal{A} \quad (4.7)$$

which is identity on objects.

Proof. Define (4.7) to be identity on objects and for any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ define the map

$$\text{Hom}_{\mathcal{A}}(E, F) \xrightarrow{(4.7)} \overline{\text{Hom}}_{\mathcal{A}}(E, F)$$

by sending any $\alpha \in \text{Hom}_{\mathcal{A}}(E, F)$ to the morphism in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ defined by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \tau} F \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\sim} F.$$

It is clear that this map is injective on morphisms and sends Id in $\mathbf{Mod}\text{-}\mathcal{A}$ to Id in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$. It remains to check that it is compatible with compositions. Let $G \in \mathbf{Mod}\text{-}\mathcal{A}$ and let $\beta \in \text{Hom}_{\mathcal{A}}(F, G)$. By definition, the images of β and α in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ under (4.7) compose into the element of $\overline{\text{Hom}}_{\mathcal{A}}(E, G)$ defined by

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \Delta} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \otimes \tau \otimes \text{Id}} F \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\beta \otimes \tau} G$$

where we suppress the isomorphisms $(-) \otimes_{\mathcal{A}} \mathcal{A} \simeq (-)$. By functoriality of tensor product, this simplifies to

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{(\beta \circ \alpha) \otimes ((\tau \otimes \tau) \circ \Delta)} G.$$

Since $(\tau \otimes \tau) \circ \Delta = \tau$, it is precisely the image of $\beta \circ \alpha$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ under (4.7). \square

The inclusion (4.7) is a special case of a more general identification which relates this section to §4.1:

Proposition 4.3. *Let \mathcal{A} be a DG-category. There is an isomorphism of DG-categories*

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\sim} (\mathbf{Nod}_{\infty}\mathcal{A})_{dg} \quad (4.8)$$

which is identity on objects. It identifies the inclusion (4.7) with $\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}^{strict} \hookrightarrow (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$.

Proof. Let $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$. The module $E \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ is isomorphic to the convolution of the twisted complex

$$\dots \longrightarrow E \otimes_k \mathcal{A}^{\otimes 3} \longrightarrow E \otimes_k \mathcal{A}^{\otimes 2} \longrightarrow E \otimes_k \mathcal{A} \quad (4.9)$$

deg.0

with the degree 0 differentials

$$E \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n+1} \xrightarrow{\sum_{i=0}^n (-1)^i \text{Id}^{\otimes i} \otimes m_2 \otimes \text{Id}^{\otimes (n-i)}} E \otimes_{\mathcal{A}} \mathcal{A}^{\otimes n}$$

where m_2 denotes either the composition map $\mathcal{A} \otimes_k \mathcal{A} \rightarrow \mathcal{A}$ or the action map $E \otimes_k \mathcal{A} \rightarrow E$, as appropriate. It follows that the elements of $\overline{\text{Hom}}_{\mathcal{A}}(E, F)$ can be identified with the twisted complex morphisms

$$\begin{array}{ccccccc} \dots & \longrightarrow & E \otimes_k \mathcal{A}^{\otimes 3} & \longrightarrow & E \otimes_k \mathcal{A}^{\otimes 2} & \longrightarrow & E \otimes_k \mathcal{A} \\ & & & & & & \downarrow \\ & & & & & & \text{deg.0} \\ & & & & & & \downarrow \\ & & & & & & F, \end{array} \quad (4.10)$$

that is — with collections $\{E \otimes_k \mathcal{A}^n \rightarrow F\}_{n \geq 1}$. Finally, \mathcal{A} -module morphisms $E \otimes_k \mathcal{A}^n \rightarrow F$ can be identified with $k_{\mathcal{A}}$ -module morphisms $E \otimes_k \mathcal{A}^{n-1} \rightarrow F$. We now define a bijective map

$$\overline{\text{Hom}}_{\mathcal{A}}(E, F) \rightarrow \text{Hom}_{\mathcal{A}}^{\infty}(E, F) \quad (4.11)$$

by sending any twisted complex morphism (4.10) to the A_{∞} -morphism defined by the corresponding $k_{\mathcal{A}}$ -module morphism collection $\{E \otimes_k \mathcal{A}^n \rightarrow F\}_{n \geq 0}$. It remains to check that the map (4.11) commutes with differentials. It is a straightforward verification.

For the last assertion, let $\alpha \in \text{Hom}_{\mathcal{A}}(E, F)$. The corresponding element of $\overline{\text{Hom}}_{\mathcal{A}}(E, F)$ is the composition of $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E$ with $E \xrightarrow{\alpha} F$. On the level of twisted complexes, the former map consists of a single component $E \otimes_k \mathcal{A} \xrightarrow{\text{act}} E$. The composition consists therefore of a single component $E \otimes_k \mathcal{A} \xrightarrow{\alpha \circ \text{act}} F$. The corresponding $k_{\mathcal{A}}$ -module morphism is $E \xrightarrow{\alpha} F$. We conclude that the resulting collection of $k_{\mathcal{A}}$ -module morphisms $\{E \otimes_k \mathcal{A}^n \rightarrow F\}_{n \geq 0}$ consists of a single non-zero component: $E \xrightarrow{\alpha} F$. This defines the strict A_{∞} -morphism $E \rightarrow F$ corresponding to α , as required. \square

In $H^0(\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ every acyclic module is isomorphic to zero. By Prop. 4.3 it is also true of $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$. As $D(\mathcal{A}) = H^0(\mathbf{Mod}\text{-}\mathcal{A})/H^0(\mathcal{A}c\mathcal{A})$ the universal property of Verdier quotient ensures that the inclusion

$$H^0(\mathbf{Mod}\text{-}\mathcal{A}) \xrightarrow{H^0((4.7))} H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$$

factors canonically as

$$H^0(\mathbf{Mod}\text{-}\mathcal{A}) \rightarrow D(\mathcal{A}) \xrightarrow{\sim} H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A}) \quad (4.12)$$

where the first morphism is the canonical projection.

The categories $\overline{\mathbf{Mod}}(-)$ are thus indeed DG-enhancements of the derived categories $D(-)$ of modules.

Lemma 4.4. *Let \mathcal{A} be a DG-category and let $E \xrightarrow{\alpha} F$ be the $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ morphism defined by a $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha'} F$.*

The category isomorphism $D(-) \xrightarrow{(4.12)} H^0(\overline{\mathbf{Mod}}(-))$ identifies α with the $D(\mathcal{A})$ morphism

$$E \xrightarrow{\text{Id} \otimes \tau^{-1}} E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha'} F. \quad (4.13)$$

Here τ^{-1} is the formal inverse of the quasi-isomorphism $\bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}$.

Proof. Since $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E$ becomes an isomorphism in $D(\mathcal{A})$, it suffices to show that its composition with (4.13) in $D(\mathcal{A})$ gets mapped to its composition with α in $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$.

By definition, the composition

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E \xrightarrow{(4.13)} F$$

is the image in $D(\mathcal{A})$ of the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism α' . Its image in $H^0(\overline{\mathbf{Mod}}\text{-}\mathcal{A})$ is therefore the image of α' under $H^0((4.7))$. In other words, it is defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha' \circ (\text{Id} \otimes \tau)} F.$$

On the other hand, the image of $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau} E$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\text{Id} \otimes \tau \otimes \tau} E.$$

Its $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ composition with α is therefore defined by the $\mathbf{Mod}\text{-}\mathcal{A}$ morphism

$$E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\alpha \circ (\text{Id} \otimes \tau \otimes \text{Id})} F.$$

The claim now follows, since the map $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\tau \otimes \text{Id} - \text{Id} \otimes \tau} \bar{\mathcal{A}}$ is null-homotopic. We write down one choice for the contracting homotopy in Lemma 4.17. \square

Corollary 4.5. *Let \mathcal{A} be a DG-category.*

A $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ morphism $E \rightarrow F$ is a homotopy equivalence if and only if the corresponding $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes_{\mathcal{A}} \bar{\mathcal{A}} \rightarrow F$ is a quasi-isomorphism.

Definition 4.6. Let \mathcal{A} and \mathcal{B} be DG-categories. Define the *bar category of bimodules* $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ as follows:

- The object set of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ is the same as that of $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$.
- For any $M, N \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ set

$$\text{Hom}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}}(M, N) = \text{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \quad (4.14)$$

and write $\overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, N)$ to denote this Hom-complex.

- Composition and the identity morphisms are defined similarly to $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

We next furnish the categories $\overline{\mathbf{Mod}}$ with adjoint bifunctors which are enhancements of the derived bifunctors $(-)^{\mathbf{L}} \otimes_{\mathcal{B}} (-)$ and $\mathbf{R} \text{Hom}_{\mathcal{B}}(-, -)$.

Definition 4.7. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Define the functor

$$\bar{\otimes}_{\mathcal{B}}: \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \otimes_{\mathcal{B}} \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \quad (4.15)$$

by setting

$$M \bar{\otimes}_{\mathcal{B}} N = M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} N \quad \forall M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}, N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}.$$

Furthermore, for any $\alpha \in \overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, M')$ and $\beta \in \overline{\text{Hom}}_{\mathcal{B}\text{-}\mathcal{C}}(N, N')$ define

$$M \bar{\otimes}_{\mathcal{B}} N \xrightarrow{\alpha \otimes \beta} M' \bar{\otimes}_{\mathcal{B}} N'$$

to be the morphism corresponding to

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta^2 \otimes \text{Id}^{\otimes 2}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\alpha \otimes \text{Id} \otimes \beta} M' \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} N'.$$

Definition 4.8. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Define the functor

$$\overline{\text{Hom}}_{\mathcal{B}}(-, -): \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \times \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \quad (4.16)$$

by setting

$$\overline{\text{Hom}}_{\mathcal{B}}(M, N) = \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \quad \forall M \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}, N \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}.$$

Furthermore, for any $\alpha \in \overline{\text{Hom}}_{\mathcal{C}\text{-}\mathcal{B}}(M', M)$ and $\beta \in \overline{\text{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(N, N')$ define

$$\overline{\text{Hom}}_{\mathcal{B}}(M, N) \xrightarrow{\beta \circ (-) \circ \alpha} \overline{\text{Hom}}_{\mathcal{B}}(M', N')$$

by the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ map

$$\begin{aligned} & \bar{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\beta_{\mathcal{A}} \otimes \text{Id} \otimes \alpha_{\mathcal{C}}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N') \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M' \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{(4.4)} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M' \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N'). \end{aligned}$$

Here $\bar{\mathcal{A}} \xrightarrow{\beta_{\mathcal{B}}} \text{Hom}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N')$ and $\bar{\mathcal{C}} \xrightarrow{\alpha_{\mathcal{C}}} \text{Hom}(M' \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M)$ are the right adjoints of the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ and $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ morphisms

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} N \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\rightarrow N', \\ \bar{\mathcal{C}} \otimes_{\mathcal{C}} M' \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\rightarrow M \end{aligned}$$

which correspond to β and α .

Definition 4.9. Let \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} be DG-categories. For any $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $L \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and $N \in \mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ define the *composition map* in $\mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$

$$\overline{\text{Hom}}_{\mathcal{B}}(M, N) \otimes_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{B}}(L, M) \xrightarrow{\text{cmps}} \overline{\text{Hom}}_{\mathcal{B}}(L, N) \quad (4.17)$$

by the corresponding $\mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ map

$$\begin{aligned} & \bar{\mathcal{D}} \otimes_{\mathcal{D}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id} \otimes \tau} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{(4.4)} \text{Hom}_{\mathcal{B}}(L \otimes_{\mathcal{B}} \bar{\mathcal{B}}, N). \end{aligned} \quad (4.18)$$

Proposition 4.10. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$. The category isomorphisms

$$D(-) \xrightarrow{(4.12)} H^0(\overline{\mathbf{Mod}}(-))$$

identify the functors $(-)^{\mathbf{L}} \otimes_{\mathcal{A}} M$ and $\mathbf{R} \text{Hom}_{\mathcal{B}}(M, -)$ with the functors $H^0((-) \otimes_{\mathcal{A}} M)$ and $H^0(\overline{\text{Hom}}_{\mathcal{B}}(M, -))$. Similarly, they identify $M \otimes_{\mathcal{B}} (-)^{\mathbf{L}}$ and $\mathbf{R} \text{Hom}_{\mathcal{A}}((M, -))$ with $H^0(M \otimes_{\mathcal{B}} (-))$ and $H^0(\overline{\text{Hom}}_{\mathcal{A}}(M, -))$.

Proof. We only prove the assertion for $(-)^{\mathbf{L}} \otimes_{\mathcal{A}} M$ and $(-) \otimes_{\mathcal{A}} M$, the others are proved similarly.

For any DG-category \mathcal{C} the following square commutes:

$$\begin{array}{ccc} \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} & \xrightarrow{(-) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M} & \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \\ (4.7) \downarrow & & \downarrow (4.7) \\ \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A} & \xrightarrow{(-) \otimes_{\mathcal{A}} M} & \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}. \end{array}$$

The factorisation (4.12) then implies that the functor $H^0((-) \otimes_{\mathcal{A}} M)$ is identified by the isomorphisms (4.12) with the functor $D(\mathcal{C}\text{-}\mathcal{A}) \rightarrow D(\mathcal{C}\text{-}\mathcal{B})$ induced by the functor

$$\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(-) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M} \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{B}.$$

On the other hand, the functor $(-)^{\mathbf{L}} \otimes_{\mathcal{A}} M$ is constructed by taking any \mathcal{A} - h -projective resolution of M . As $(-) \otimes_{\mathcal{A}} \bar{\mathcal{A}}$ is a functorial semifree, and thus h -projective, resolution, the functor $(-)^{\mathbf{L}} \otimes_{\mathcal{A}} M$ is also isomorphic to the functor $D(\mathcal{C}\text{-}\mathcal{A}) \rightarrow D(\mathcal{C}\text{-}\mathcal{B})$ induced by $(-) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M$. \square

Proposition 4.11. Let \mathcal{A} and \mathcal{B} be DG-categories. The category isomorphisms $\overline{\mathbf{Mod}}(-) \xrightarrow{\sim} (\mathbf{Mod}_{\infty})_{dg}(-)$ of Prop. 4.3 identify the bifunctors $(-) \otimes_{\mathcal{B}} (-)$, $\overline{\text{Hom}}_{\mathcal{A}}(-, -)$, and $\overline{\text{Hom}}_{\mathcal{B}}(-, -)$ with the bifunctors $(-) \otimes_{\mathcal{B}}^{\infty} (-)$, $\text{Hom}_{\mathcal{B}}(-, -)$, and $\text{Hom}_{\mathcal{A}}(-, -)$.

Proof. Straightforward verification. \square

In view of Propositions 4.11 and 4.3 we could deduce the Tensor-Hom adjunction for $\overline{\mathbf{Mod}}$ from the Tensor-Hom adjunction for A_{∞} -modules [LH03, Lemme 4.1.1.4]. However, it is more convenient to prove this adjunction directly in $\overline{\mathbf{Mod}}$ by exhibiting explicit formulas for its unit and counit:

Proposition 4.12. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

- (1) The functors $(-) \otimes_{\mathcal{A}} M$ and $\overline{\text{Hom}}_{\mathcal{B}}(M, -)$ are left and right adjoint functors $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A} \leftrightarrow \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$. The unit and the counit of the adjunction are the maps

$$E \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{B}}(M, E \otimes_{\mathcal{A}} M) \quad \forall E \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}, \quad (4.19)$$

$$\overline{\text{Hom}}_{\mathcal{B}}(M, F) \otimes_{\mathcal{A}} M \xrightarrow{\text{ev}} F \quad \forall F \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B} \quad (4.20)$$

in $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and $\mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$ which correspond to the $\mathcal{C}\text{-Mod}\text{-}\mathcal{A}$ and $\mathcal{C}\text{-Mod}\text{-}\mathcal{B}$ maps

$$\bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \xrightarrow{\tau \otimes \text{mlt}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B}) \xrightarrow{(\text{Id}^{\otimes 3} \otimes \tau) \circ (-)} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M), \quad (4.21)$$

$$\bar{C} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, F) \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id}^{\otimes 2}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, F) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\text{ev}} F. \quad (4.22)$$

- (2) The functors $M \otimes_{\mathcal{B}} (-)$ and $\overline{\text{Hom}}_{\mathcal{A}}(M, -)$ are left and right adjoint functors $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C} \leftrightarrow \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$. The unit and the counit of the adjunction are given by the maps

$$E \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{A}}(M, M \otimes_{\mathcal{B}} E) \quad \forall E \in \mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}, \quad (4.23)$$

$$M \otimes_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, F) \xrightarrow{\text{ev}} F \quad \forall F \in \mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C} \quad (4.24)$$

in $\mathcal{B}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$ and $\mathcal{A}\text{-}\overline{\text{Mod}}\text{-}\mathcal{C}$ which correspond to the in $\mathcal{B}\text{-Mod}\text{-}\mathcal{C}$ and $\mathcal{A}\text{-Mod}\text{-}\mathcal{C}$ maps

$$\bar{B} \otimes_{\mathcal{B}} E \otimes_{\mathcal{C}} \bar{C} \xrightarrow{\text{mlt} \otimes \tau} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \otimes_{\mathcal{B}} E), \xrightarrow{(\tau \otimes \text{Id}^{\otimes 3}) \circ (-)} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, M \otimes_{\mathcal{B}} \bar{B} \otimes_{\mathcal{B}} E), \quad (4.25)$$

$$\bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, F) \otimes_{\mathcal{C}} \bar{C} \xrightarrow{\text{Id}^{\otimes 2} \otimes \tau \otimes \text{Id} \otimes \tau} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \text{Hom}_{\mathcal{A}}(\bar{A} \otimes_{\mathcal{A}} M, F) \xrightarrow{\text{ev}} F. \quad (4.26)$$

Proof. To prove the assertion (1) it suffices to show that for any $E \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{A}$ and $F \in \mathcal{C}\text{-}\overline{\text{Mod}}\text{-}\mathcal{B}$

$$E \otimes_{\mathcal{A}} M \xrightarrow{\text{mlt} \otimes \text{Id}} \overline{\text{Hom}}_{\mathcal{B}}(M, E \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} M \xrightarrow{\text{ev}} E \otimes_{\mathcal{A}} M \quad (4.27)$$

$$\overline{\text{Hom}}_{\mathcal{B}}(M, F) \xrightarrow{\text{mlt}} \overline{\text{Hom}}_{\mathcal{B}}(M, \overline{\text{Hom}}_{\mathcal{B}}(M, F) \otimes_{\mathcal{A}} M) \xrightarrow{\text{ev} \circ (-)} \overline{\text{Hom}}_{\mathcal{B}}(M, F) \quad (4.28)$$

are identity morphisms. We only demonstrate this for (4.27), as (4.28) works out very similarly.

By definition of composition in $\overline{\text{Mod}}\text{-}\mathcal{B}$, (4.27) corresponds to the $\text{Mod}\text{-}\mathcal{B}$ map

$$\begin{aligned} & \bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\Delta \otimes \text{Id} \otimes \Delta \otimes \text{Id}^{\otimes 2}} \bar{C} \otimes_{\mathcal{C}} \bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\text{Id} \otimes \tau \otimes \text{mlt} \otimes \text{Id}^{\otimes 3}} \\ & \rightarrow \bar{C} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B}) \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\text{Id} \otimes ((\tau \otimes \text{Id}^{\otimes 3}) \circ (-)) \otimes \text{Id}^{\otimes 3}} \\ & \rightarrow \bar{C} \otimes_{\mathcal{C}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\tau \otimes \text{Id} \otimes \tau \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\text{ev}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M. \end{aligned}$$

By functoriality of the tensor product the above composition equals

$$\begin{aligned} & \bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\Delta \otimes \text{Id} \otimes \Delta \otimes \text{Id}^{\otimes 2}} \bar{C} \otimes_{\mathcal{C}} \bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\tau \otimes \text{Id}^{\otimes 3} \otimes \tau \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \bar{C} \otimes_{\mathcal{C}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\tau \otimes \text{mlt} \otimes \text{Id}^{\otimes 2}} \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B}) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{((\text{Id}^{\otimes 3} \otimes \tau) \circ (-)) \otimes \text{Id}^{\otimes 2}} \\ & \rightarrow \text{Hom}_{\mathcal{B}}(M \otimes_{\mathcal{B}} \bar{B}, E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{B} \xrightarrow{\text{ev}} E \otimes_{\mathcal{A}} \bar{A} \otimes_{\mathcal{A}} M. \end{aligned}$$

Since $\tau \circ \Delta = \text{Id}$, the first two maps compose to Id . On the other hand, the last two maps compose to $(\text{Id}^{\otimes 3} \otimes \tau) \circ \text{ev}$. By the Tensor-Hom adjunction for $M \otimes_{\mathcal{B}} \bar{B}$ the total composition is therefore $\tau \otimes \text{Id}^{\otimes 3} \otimes \tau$. The corresponding map in $\overline{\text{Mod}}\text{-}\mathcal{B}$ is Id , as desired.

The assertion (2) is settled similarly. \square

It is worth writing out the maps defining the units and the counits of these two adjunctions explicitly. The compositions (4.21), (4.22), (4.25), and (4.26) are the maps

$$\begin{aligned} & c \otimes e \otimes a \mapsto (\tau(c)e \otimes a \otimes -) \circ (\text{Id} \otimes \tau) \\ & c \otimes \alpha \otimes a \otimes m \otimes b \mapsto \tau(c)\alpha(\tau(a).m \otimes b) \\ & b \otimes e \otimes c \mapsto \left((-1)^{\deg(-)(\deg(b)+\deg(e))} (-) \otimes b \otimes e\tau(c) \right) \circ (\tau \otimes \text{Id}) \\ & a \otimes m \otimes b \otimes \alpha \otimes c \mapsto (-1)^{\deg(\alpha)(\deg(a)+\deg(m)+\deg(b))} \alpha(a \otimes m.\tau(b))\tau(c). \end{aligned}$$

To sum up, we have a DG-enhancement framework which to every DG-category \mathcal{A} associates an enhancement $\overline{\text{Mod}}\text{-}\mathcal{A}$ of its derived category $D(\mathcal{A})$. These enhancements $\overline{\text{Mod}}$ admit genuinely adjoint (in each argument) bifunctors $(-) \otimes_{\bullet} (-)$ and $\overline{\text{Hom}}_{\bullet}(-, -)$.

4.3. **On non-invertibility of the semifree resolution $\mathcal{A} \overline{\otimes} M \rightarrow M$.** Recall the semifree resolution

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{(3.37)} M$$

discussed in §3.5. Consider moreover its right adjoint

$$M \longrightarrow \mathrm{Hom}_{\mathcal{A}}^{\infty}(\mathcal{A}, M)$$

with respect to $\mathcal{A} \overline{\otimes}_{\mathcal{A}}(-)$.

The category isomorphism $(\mathcal{A}\text{-}\mathbf{Nod}_{\infty}\text{-}\mathcal{B})_{dg} \simeq \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ of Prop. 4.3 identifies these with the maps

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \longrightarrow M \tag{4.29}$$

$$M \longrightarrow \overline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, M) \tag{4.30}$$

defined by the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\tau \otimes \tau \otimes \mathrm{Id} \otimes \tau} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\tau \otimes \mathrm{Id} \otimes \tau} M \xrightarrow{(2.9)} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{(-) \circ \tau} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M). \end{aligned}$$

We therefore see that the $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ maps (4.29) and (4.30) are the analogues of the canonical $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ isomorphisms $\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{(2.7)} M$ and $M \xrightarrow{(2.9)} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M)$. Indeed, they induce the same isomorphisms $\mathcal{A} \otimes_{\mathcal{A}}^{\mathbf{L}} M \simeq M$ and $M \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M)$ in the derived category $D(\mathcal{A}\text{-}\mathcal{B})$ as (2.7) and (2.9).

The biggest drawback of the categories $\overline{\mathbf{Mod}}$ is that the maps (4.29) and (4.30) are not themselves isomorphisms, like (2.7) and (2.9), but merely homotopy equivalences.

In this section, we show that this can be controlled. The maps (4.29) and (4.30) have natural semi-inverses. These are genuine inverses on one side, but only homotopy inverses on the other. However, the arising higher homotopies are induced by endomorphisms of the bar complex and thus independent of M .

To put this into context, recall [Dri04, §3.7], [Tab05], [AL13, Appendix A] that for any DG-category \mathcal{A} and any objects $x, y \in \mathcal{A}$ we can (non-canonically) complete any homotopy equivalence

$$x \xrightarrow{\beta} y \tag{4.31}$$

to the following system of morphisms and relations between them. The dotted arrows denote the morphisms of degree -1 and the dashed arrow the morphism of degree -2 :

$$\begin{aligned} d\theta_x &= \alpha \circ \beta - \mathrm{Id}_x, \\ d\theta_y &= \mathrm{Id}_y - \beta \circ \alpha, \\ d\alpha &= d\beta = 0, \\ d\phi &= -\beta \circ \theta_x - \theta_y \circ \beta. \end{aligned} \quad \begin{array}{c} \phi \\ \text{---} \\ \beta \\ \text{---} \\ \alpha \end{array} \quad \begin{array}{c} \theta_x \\ \text{---} \\ x \\ \text{---} \\ y \\ \text{---} \\ \theta_y \end{array} \tag{4.32}$$

In other words, we can find:

- a homotopy inverse α of β ,
- a degree -1 homotopy θ_x from $\beta \circ \alpha$ to Id_x
- a degree -1 homotopy θ_y from $\alpha \circ \beta$ to Id_y
- a degree -2 homotopy ϕ from $\beta \circ \theta_x$ to $\theta_y \circ \beta$.

The key assertion here is that we can choose θ_x and θ_y so that ϕ exists.

It turns out that in the case of homotopy equivalences (4.29) and (4.30) we can do quite a bit better than (4.32). Firstly, they admit natural one-sided inverses:

Definition 4.13. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Define the maps

$$M \longrightarrow \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \tag{4.33}$$

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, M) \longrightarrow M \tag{4.34}$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\mathrm{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\mathrm{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \xrightarrow{\mathrm{ev}} M. \end{aligned}$$

It can be readily checked that (4.33) is a left inverse to (4.29), while (4.34) is a right inverse to (4.30). We can apply these to give a more natural description of Tensor-Hom adjunction counits, and to show action maps to be instances of Tensor-Hom adjunction units:

Lemma 4.14. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.*

The composition

$$\mathrm{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{Id} \otimes (4.30)} \mathrm{Hom}_{\mathcal{B}}(M, -) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, M) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, -) \xrightarrow{(4.34)} \mathrm{Id}_{\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}} \quad (4.35)$$

is the counit of the $((-) \otimes_{\mathcal{A}} M, \mathrm{Hom}_{\mathcal{B}}(M, -))$ adjunction. The counit of the $(M \otimes_{\mathcal{B}} (-), \mathrm{Hom}_{\mathcal{A}}(M, -))$ adjunction admits an analogous description.

Proof. Direct verification. □

Lemma 4.15. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. The compositions*

$$\begin{aligned} \mathcal{A} &\xrightarrow{\mathrm{mlt}} \mathrm{Hom}_{\mathcal{B}}(M, \mathcal{A} \otimes_{\mathcal{A}} M) \xrightarrow{(4.29)} \mathrm{Hom}_{\mathcal{B}}(M, M) \\ \mathcal{B} &\xrightarrow{\mathrm{mlt}} \mathrm{Hom}_{\mathcal{A}}(M, M \otimes_{\mathcal{B}} \mathcal{B}) \xrightarrow{(4.29)} \mathrm{Hom}_{\mathcal{A}}(M, M) \end{aligned}$$

are the maps

$$\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(M, M) \quad (4.36)$$

$$\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M, M) \quad (4.37)$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ induced by the corresponding action maps.

Proof. Direct verification. □

In order to define degree -1 and -2 homotopies as per (4.32) we need to introduce certain natural endomorphisms of the bar complex. Let \mathcal{A} be a DG-category. Recall that its bar complex $\bar{\mathcal{A}}$ carries a natural structure of a coalgebra in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ with both the comultiplication Δ and the counit being homotopy equivalences. Since $\tau \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes \tau$ are both right inverses to Δ , the following morphism is a boundary:

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\tau \otimes \mathrm{Id} - \mathrm{Id} \otimes \tau} \bar{\mathcal{A}}. \quad (4.38)$$

Definition 4.16. Let \mathcal{A} be a DG-category. Define the degree -1 map

$$\mu: \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \quad (4.39)$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ by

$$(a_1 \otimes \cdots \otimes a_i) \otimes_{\mathcal{A}} (a_{i+1} \otimes \cdots \otimes a_n) \mapsto (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n. \quad (4.40)$$

Lemma 4.17. *Let \mathcal{A} be a DG-category. Then*

$$d\mu = \tau \otimes \mathrm{Id} - \mathrm{Id} \otimes \tau$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Proof. A direct computation. By definition the map $d\mu$ consists of two components:

$$d\mu(-) = d_{\bar{\mathcal{A}}}(\mu(-)) + \mu(d_{\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}}(-)).$$

The first component always sends each

$$(a_1 \otimes \cdots \otimes a_i) \otimes_{\mathcal{A}} (a_{i+1} \otimes \cdots \otimes a_n) \in \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \quad (4.41)$$

to

$$\begin{aligned} &d_{\bar{\mathcal{A}}}((-1)^i a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n) = \\ &= \sum_{j=1}^{i-2} (-1)^{i+j-1} a_1 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \quad + \\ &+ a_1 \otimes \cdots \otimes (a_{i-1} a_i a_{i+1}) \otimes \cdots \otimes a_n \quad - \\ &- a_1 \otimes \cdots \otimes (a_i a_{i+1} a_{i+2}) \otimes \cdots \otimes a_n \quad + \\ &+ \sum_{j=i+2}^{n-1} (-1)^{i+j} a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes a_n. \end{aligned} \quad (4.42)$$

On the other hand, the second component sends each (4.41) to

$$\mu(d_{\bar{\mathcal{A}}}(a_1 \otimes \cdots \otimes a_i) \otimes_{\mathcal{A}} (a_{i+1} \otimes \cdots \otimes a_n) + (-1)^{i-2} (a_1 \otimes \cdots \otimes a_i) \otimes_{\mathcal{A}} d_{\bar{\mathcal{A}}}(a_{i+1} \otimes \cdots \otimes a_n)). \quad (4.43)$$

The first summand of (4.43) is 0 when $i = 2$ and for $i > 2$ it equals

$$\begin{aligned} & \mu \left(\sum_{j=1}^{i-1} (-1)^{j-1} (a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i) \otimes_{\mathcal{A}} (a_{i+1} \otimes \cdots \otimes a_n) \right) = \\ &= \sum_{j=1}^{i-2} (-1)^{i+j} a_1 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \quad - \\ & - a_1 \otimes \cdots \otimes (a_{i-1} a_i a_{i+1}) \otimes \cdots \otimes a_n \end{aligned}$$

and thus cancels the first two summands of (4.42). Similarly, the second summand of (4.43) is 0 when $n-i = 2$ and cancels out the other two summands of (4.42) when $n-i > 2$.

Thus, (4.43) cancels out all of (4.42) except for $a_1 a_2 a_3 \otimes a_4 \otimes \cdots \otimes a_n$ when $i = 2$ and $a_1 \otimes \cdots \otimes a_{n-3} \otimes a_{n-2} a_{n-1} a_n$ when $n-i = 2$. In other words, (4.42) + (4.43) is precisely $\tau \otimes \text{Id} + \text{Id} \otimes \tau$ applied to (4.41), as required. \square

We next look at the compositions of this lift μ with the projection τ and the comultiplication Δ .

Definition 4.18. Let \mathcal{A} be a DG-category and let $k \in \mathbb{Z}_{\geq 0}$. Define the *insertion of k 1s* map

$$\lambda_k: \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}} \quad (4.44)$$

to be the degree $-k$ map which sends any $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}^{\otimes n}$ to

$$\sum_{i_1+i_2+\cdots+i_{k+1}=n} (-1)^{ki_1+(k-1)i_2+\cdots+1i_k+k} a_1 \otimes \cdots \otimes a_{i_1} \otimes 1 \otimes a_{i_1+1} \otimes \cdots \otimes a_{i_1+i_2} \otimes 1 \otimes \cdots \otimes 1 \otimes a_{i_1+\cdots+i_k+1} \otimes \cdots \otimes a_n.$$

Recall that as Δ is the comultiplication of the coalgebra $\bar{\mathcal{A}}$ it is coassociative. We therefore write Δ^k for the unique map $\bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}^{\otimes(k+1)}$ which is a composition of k applications of Δ .

Proposition 4.19. (1) *The composition $\bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}} \xrightarrow{\tau} \mathcal{A}$ is 0.*

(2) *The composition $\bar{\mathcal{A}} \xrightarrow{\Delta} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}}$ is the map λ_1 .*

(3) *For any $k \geq 0$ the map λ_k equals the composition*

$$\bar{\mathcal{A}} \xrightarrow{\Delta^k} \bar{\mathcal{A}}^{\otimes(k+1)} \xrightarrow{\text{Id}^{\otimes(k-1)} \otimes \mu} \bar{\mathcal{A}}^{\otimes k} \xrightarrow{\text{Id}^{\otimes(k-2)} \otimes \mu} \bar{\mathcal{A}}^{\otimes(k-1)} \rightarrow \cdots \rightarrow \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\mu} \bar{\mathcal{A}}. \quad (4.45)$$

(4) *For any $k \geq 1$ the map λ_k equals the composition $\lambda_k = \mu \circ (\text{Id} \otimes \lambda_{k-1}) \circ \Delta$.*

(5) *For any $k \geq 1$ the map $d\lambda_k$ equals λ_{k-1} if k is even, and 0 if k is odd.*

Proof. (1): By definition, τ sends each $a_1 \otimes \cdots \otimes a_n \in \bar{\mathcal{A}}$ to 0 for $n > 2$. On the other hand, by definition of μ its image in $\bar{\mathcal{A}}$ is spanned by $a_1 \otimes \cdots \otimes a_n \in \bar{\mathcal{A}}$ with $n > 2$. It follows that $\tau \circ \mu = 0$.

(2): For any $a_1 \otimes \cdots \otimes a_n \in \bar{\mathcal{A}}$ we have

$$\begin{aligned} & \mu(\Delta(a_1 \otimes \cdots \otimes a_n)) = \\ &= \mu \left(\sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes_{\mathcal{A}} (1 \otimes a_{i+1} \otimes \cdots \otimes a_n) \right) = \\ &= \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n = \lambda_1(a_1 \otimes \cdots \otimes a_n). \end{aligned}$$

(3): This follows by a direct computation analogous to the one for (2).

(4): By the assertion (3) we have

$$\begin{aligned} \lambda_k &= \mu \circ (\text{Id} \otimes \mu) \circ \cdots \circ (\text{Id}^{\otimes(k-1)} \otimes \mu) \circ \Delta^k = \\ &= \mu \circ (\text{Id} \otimes \mu) \circ \cdots \circ (\text{Id}^{\otimes(k-1)} \otimes \mu) \circ (\text{Id}^{\otimes(k-1)} \otimes \Delta) \circ \cdots \circ (\text{Id} \otimes \Delta) \circ \Delta = \\ &= \mu \circ \left(\text{Id} \otimes \left(\mu \circ \cdots \circ (\text{Id}^{\otimes(k-2)} \otimes \mu) \circ (\text{Id}^{\otimes(k-2)} \otimes \Delta) \circ \cdots \circ \Delta \right) \right) \circ \Delta = \\ &= \mu \circ (\text{Id} \otimes \lambda_{k-1}) \circ \Delta. \end{aligned}$$

$$d\lambda_1 = d(\mu \circ \Delta) = (d\mu) \circ \Delta = (\tau \otimes \text{Id} - \text{Id} \otimes \tau) \circ \Delta = \text{Id} - \text{Id} = 0.$$
$$\begin{aligned} d\lambda_{n+1} &= d(\mu \circ (\text{Id} \otimes \lambda_n) \circ \Delta) = d\mu \circ (\text{Id} \otimes \lambda_n) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta = \\ &= (\tau \otimes \text{Id} - \text{Id} \otimes \tau) \circ (\text{Id} \otimes (\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta)) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta = \\ &= (\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta) \circ (\tau \otimes \text{Id}) \circ \Delta - (\text{Id} \otimes (\tau \circ \mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta)) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta. \end{aligned}$$
$$\mu \circ (\text{Id} \otimes \lambda_{n-1}) \circ \Delta - \mu \circ (\text{Id} \otimes d\lambda_n) \circ \Delta.$$

Having established these properties of the maps μ and λ_k , we can now proceed to our main objective:

$$M \xrightarrow{\beta_k} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \quad (4.46)$$

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\theta} \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \quad (4.47)$$

$$\mathcal{A} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\alpha} M \quad (4.48)$$

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\lambda_k \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\mu \text{Id} \otimes \tau} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\tau \otimes \tau \text{Id} \otimes \tau} M. \end{aligned}$$

- (1) $d\theta = \text{Id} - \beta_0 \circ \alpha$.
- (2) $0 = \alpha \circ \beta_0 - \text{Id}$.
- (3) $\beta_k = \theta^k \circ \beta_0$.
- (4) For any $k \geq 1$ we have $\alpha \circ \beta_k = 0$.
- (5) For any $k \geq 1$ the map $d\beta_k$ equals 0 when k is odd, and β_{k-1} when k is even.

$$d\theta = d(\mu \otimes \text{Id} \otimes \tau) = (d\mu) \otimes \text{Id} \otimes \tau = \tau \otimes \text{Id} \otimes \text{Id} \otimes \tau - \text{Id} \otimes \tau \otimes \text{Id} \otimes \tau.$$
$$\bar{A} \otimes_{\mathcal{A}} (\bar{A} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\tau \otimes \text{Id} \otimes \text{Id} \otimes \tau} \bar{A} \otimes_{\mathcal{A}} M$$

Proposition 4.22. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Let α , β_0 , and θ be the maps, respectively, introduced in Definition 4.20:*

$$\begin{array}{ccc}
M & \xrightarrow{\beta_0} & \mathcal{A} \otimes_{\mathcal{A}} M \\
& \searrow \alpha & \uparrow \theta \\
& & \mathcal{A} \otimes_{\mathcal{A}} M
\end{array}
\quad (4.49)$$

The sub-DG-category of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ generated by α, β_0 and θ is the free DG-category generated by those elements modulo the following relations:

- (1) $d\alpha = d\beta_0 = 0$,
- (2) $d\theta = \text{Id} - \beta_0 \circ \alpha$,
- (3) $0 = \alpha \circ \beta_0 - \text{Id}$,
- (4) $\alpha \circ \theta = 0$.

NB: The relations in Prop. 4.22 can be obtained by taking those in (4.32) and demanding further that $\theta_x = 0$, $\alpha \circ \theta_y = 0$, and $\phi = \theta_y^2 \circ \beta$.

Proof. It follows from Prop. 4.21 that the relations (1)-(4) do hold. It remains to show that no other relations are necessary. This is equivalent to showing that:

- (1) $\theta^k \beta$ for $k \geq 0$ are distinct elements of $\text{Hom}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}}(M, \mathcal{A} \otimes_{\mathcal{A}} M)$.
- (2) $\theta^k, \theta^k \circ \beta \circ \alpha$ for all $k \geq 0$ are distinct elements of $\text{Hom}_{\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}}(\mathcal{A} \otimes_{\mathcal{A}} M, \mathcal{A} \otimes_{\mathcal{A}} M)$.

For (1) it is enough to note that each $\theta^k \beta$ is of degree $-k$. We can, however, describe it explicitly. By Prop. 4.21 we have $\theta^k \circ \beta = \beta_k$. Thus it is induced by the map $\bar{\mathcal{A}} \xrightarrow{\lambda_k} \bar{\mathcal{A}}$ of Definition 4.18 which inserts k 1s into an element of $\bar{\mathcal{A}}$.

For (2), we similarly see that for each $k \geq 1$ the maps θ^k and $\theta^k \circ \beta_0 \circ \alpha$ are induced by the maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} &\xrightarrow{(-1)^{k-1} \mu \circ (\lambda_{k-1} \otimes \text{Id})} \bar{\mathcal{A}}, \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}} &\xrightarrow{\lambda_k \otimes \tau} \bar{\mathcal{A}}, \end{aligned}$$

respectively. These are evidently two different maps. □

Similarly, we have:

Definition 4.23. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Define the maps

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{\delta_k} M \tag{4.50}$$

$$\text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \xrightarrow{\kappa} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \tag{4.51}$$

$$M \xrightarrow{\gamma} \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) \tag{4.52}$$

of degree $-k$, -1 , and 0 , respectively, in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by the corresponding $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$\begin{aligned} \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\text{ev} \circ (\lambda_k \otimes \text{Id} \otimes \tau)} M \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (\text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M)) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{\text{cmps} \circ (((\mu \circ (-)) \circ \text{mlt}) \otimes \text{Id} \otimes \tau)} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \\ \bar{\mathcal{A}} \otimes_{\mathcal{A}} (M) \otimes_{\mathcal{B}} \bar{\mathcal{B}} &\xrightarrow{((-) \circ \tau) \circ (\tau \otimes (2.9) \otimes \tau)} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, M) \end{aligned}$$

where $(\mu \circ (-)) \circ \text{mlt}$ denotes the composition $\bar{\mathcal{A}} \xrightarrow{\text{mlt}} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \bar{\mathcal{A}} \otimes_{\mathcal{A}} \bar{\mathcal{A}}) \xrightarrow{\mu \circ (-)} \text{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \bar{\mathcal{A}})$.

NB: The map γ is the canonical map (4.30), while δ_0 is its left inverse (4.34).

Proposition 4.24. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. We have:

- (1) $d\kappa = \gamma \circ \delta_0 - \text{Id}$.
- (2) $0 = \text{Id} - \delta_0 \circ \gamma$.
- (3) $\delta_k = \delta_0 \circ \kappa^k$.
- (4) For any $k \geq 1$ we have $\delta_k \circ \gamma = 0$.
- (5) For any $k \geq 1$ the map $d\delta_k$ equals 0 when k is odd, and δ_{k-1} when k is even.

Proof. Analogous to the proof of Prop. 4.21. □

Proposition 4.25. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Let γ , δ_0 , and κ be the maps, respectively, introduced in Definition 4.23:

$$\begin{array}{ccc} & \delta_0 & \\ & \curvearrowright & \\ \kappa \circ \text{Id} & \text{Hom}_{\mathcal{A}}(\mathcal{A}, M) & \xrightarrow{\delta_0} M \\ & \curvearrowleft & \\ & \gamma & \end{array} \tag{4.53}$$

The sub-DG-category of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ generated by γ , δ_0 , and κ is the free DG-category generated by those elements modulo the following relations:

- (1) $d\gamma = d\delta_0 = 0$,
- (2) $d\kappa = \gamma \circ \delta_0 - \text{Id}$,
- (3) $0 = \text{Id} - \delta_0 \circ \gamma$,
- (4) $\kappa \circ \gamma = 0$.

Proof. Analogous to the proof of Prop. 4.22. \square

Finally, the results in this section have been so far concerned with the left action of \mathcal{A} on M . They each have a counterpart for the right action of \mathcal{B} of M . We leave this as an exercise for the reader.

4.4. Dualisation. In this section we look at the dualising functors for bar categories of bimodules:

Definition 4.26. Let \mathcal{A} and \mathcal{B} be DG-categories. Define the *dualising functors* $(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\text{opp}} \rightarrow \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$

$$\begin{aligned} (-)^{\bar{\mathcal{A}}} &\stackrel{\text{def}}{=} \overline{\text{Hom}}_{\mathcal{A}}(-, \mathcal{A}) \\ (-)^{\bar{\mathcal{B}}} &\stackrel{\text{def}}{=} \overline{\text{Hom}}_{\mathcal{B}}(-, \mathcal{B}). \end{aligned}$$

Definition 4.27. Let \mathcal{A} and \mathcal{B} be DG-categories. Define the natural transformation

$$\text{Id} \longrightarrow (-)^{\bar{\mathcal{A}}\bar{\mathcal{A}}} \quad (4.54)$$

of endofunctors of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ by setting for every $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ the corresponding map

$$M \longrightarrow \overline{\text{Hom}}_{\mathcal{A}}(\overline{\text{Hom}}_{\mathcal{A}}(M, \mathcal{A}), \mathcal{A})$$

to be the right adjoint of the evaluation map

$$M \otimes_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, \mathcal{A}) \xrightarrow{\text{ev}} \mathcal{A}$$

with respect to the functor $(-)^{\bar{\mathcal{B}}}\overline{\text{Hom}}_{\mathcal{A}}(M, \mathcal{A})$.

Define the natural transformation

$$\text{Id} \longrightarrow (-)^{\bar{\mathcal{B}}\bar{\mathcal{B}}} \quad (4.55)$$

similarly.

Lemma 4.28. Let \mathcal{A} and \mathcal{B} be DG-categories.

- (1) The natural transformations (4.54) and (4.55) are homotopy equivalences on \mathcal{A} - and \mathcal{B} -perfect bimodules, respectively.
- (2) The functors $(-)^{\bar{\mathcal{A}}}$ and $(-)^{\bar{\mathcal{B}}}$ restrict to quasi-equivalences

$$((\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\mathcal{A}\text{-}\mathcal{P}erf})^{\text{opp}} \xrightarrow{(-)^{\bar{\mathcal{A}}}} (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\mathcal{A}\text{-}\mathcal{P}erf}, \quad (4.56)$$

$$((\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})^{\mathcal{B}\text{-}\mathcal{P}erf})^{\text{opp}} \xrightarrow{(-)^{\bar{\mathcal{B}}}} (\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A})^{\mathcal{B}\text{-}\mathcal{P}erf}, \quad (4.57)$$

Proof. (1): We proceed by reduction to a similar result for ordinary categories of DG bimodules proved in [AL13, §2]. Let $M \in \overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and let $a \in \mathcal{A}$. Since $(M^{\bar{\mathcal{B}}})_a = ({}_a M)^{\bar{\mathcal{B}}}$, it is clear that the fiber over a of

$$M \xrightarrow{(4.55)} M^{\bar{\mathcal{B}}\bar{\mathcal{B}}} \quad (4.58)$$

is the analogous natural transformation (4.55) of endofunctors of $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ applied to ${}_a M$.

It follows from the description of the adjunction unit (4.23) that the $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ map

$${}_a M \xrightarrow{(4.55)} ({}_a M)^{\bar{\mathcal{B}}\bar{\mathcal{B}}}$$

is defined by the $\mathbf{Mod}\text{-}\mathcal{B}$ map

$$\begin{aligned} &{}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \xrightarrow{\text{mlt}} \text{Hom}_{\mathcal{B}}\left(({}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\bar{\mathcal{B}}}, {}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} ({}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\bar{\mathcal{B}}}\right) \xrightarrow{\text{ev} \circ (-)} \\ &\rightarrow \text{Hom}_{\mathcal{B}}\left(({}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\bar{\mathcal{B}}}, \mathcal{B}\right) \xrightarrow{(-) \circ (\text{Id} \otimes \tau)} \text{Hom}_{\mathcal{B}}\left(({}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\bar{\mathcal{B}}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, \mathcal{B}\right). \end{aligned} \quad (4.59)$$

If M is \mathcal{B} -perfect, ${}_a M$ is perfect. Therefore ${}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}}$ is h -projective and perfect, and hence so is $({}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}})^{\bar{\mathcal{B}}}$. Thus the last map in the composition above is a quasi-isomorphism. On the other hand, the first two maps define a natural transformation $\text{Id} \rightarrow (-)^{\bar{\mathcal{B}}\bar{\mathcal{B}}}$ of endofunctors of \mathcal{B} . It is a quasi-isomorphism on representables,

and hence on all h -projective and perfect modules, cf. [AL13, §2]. In particular, it is a quasi-isomorphism on ${}_a M \otimes_{\mathcal{B}} \bar{\mathcal{B}}$. Thus (4.59) is a quasi-isomorphism.

We conclude that for a \mathcal{B} -perfect M the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ map which defines (4.58) is a quasi-isomorphism, since its every fiber over \mathcal{A} is. It follows from Cor. 4.5 that (4.58) itself is a homotopy equivalence, as desired.

A similar argument shows that (4.54) is a homotopy equivalence on \mathcal{A} -perfect bimodules.

(2): By the Tensor-Hom adjunction we have for any $M, N \in \overline{\mathbf{Mod}}\text{-}\mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}}(M^{\bar{\mathcal{A}}}, N^{\bar{\mathcal{A}}}) \simeq \mathrm{Hom}_{\mathcal{A}}(N, M^{\bar{\mathcal{A}}\bar{\mathcal{A}}}),$$

and it can be readily checked that this isomorphism identifies

$$\mathrm{Hom}_{\mathcal{A}}(N, M) \xrightarrow{(-)^{\bar{\mathcal{A}}}} \mathrm{Hom}_{\mathcal{A}}(M^{\bar{\mathcal{A}}}, N^{\bar{\mathcal{A}}})$$

with

$$\mathrm{Hom}_{\mathcal{A}}(N, M) \xrightarrow{(4.55) \circ (-)} \mathrm{Hom}_{\mathcal{A}}(N, M^{\bar{\mathcal{A}}\bar{\mathcal{A}}}).$$

The claim for $(-)^{\bar{\mathcal{A}}}$ now follows from (1). The claim for $(-)^{\bar{\mathcal{B}}}$ is proved similarly. \square

The following is the $\overline{\mathbf{Mod}}$ analogue of the map (2.13):

Definition 4.29. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be DG-categories. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $N \in \mathcal{D}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and $L \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Define the $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ map

$$L \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M) \quad (4.60)$$

as the composition

$$L \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\mathrm{mlt} \otimes \mathrm{Id}} \mathrm{Hom}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M).$$

Lemma 4.30. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be DG-categories and let $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, $N \in \mathcal{D}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$, and $L \in \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$. The $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{D}$ map

$$L \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N, M) \xrightarrow{(4.60)} \mathrm{Hom}_{\mathcal{B}}(N, L \otimes_{\mathcal{A}} M)$$

is a homotopy equivalence when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect.

Proof. It follows from the definitions of the adjunction unit (4.23) and the composition map (4.17) that the $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{D}$ map defining (4.60) is

$$\begin{aligned} & L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{\mathrm{mlt} \otimes \mathrm{Id}} \mathrm{Hom}_{\mathcal{B}}(M, L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, M) \xrightarrow{\mathrm{cmps}} \\ & \rightarrow \mathrm{Hom}_{\mathcal{B}}(N \otimes_{\mathcal{B}} \bar{\mathcal{B}}, L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \otimes_{\mathcal{A}} M). \end{aligned}$$

Thus it is an instance of the map (2.13) and is therefore a quasi-isomorphism when $N \otimes_{\mathcal{B}} \bar{\mathcal{B}} \in \mathcal{P}^{\mathcal{B}\text{-}\mathcal{P}^{erf}}(\mathcal{A}\text{-}\mathcal{B})$ or $L \otimes_{\mathcal{A}} \bar{\mathcal{A}} \in \mathcal{P}^{\mathcal{A}\text{-}\mathcal{P}^{erf}}(\mathcal{A}\text{-}\mathcal{B})$. Hence when N is \mathcal{B} -perfect or L is \mathcal{A} -perfect the $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{D}$ map defining (4.60) is a quasi-isomorphism, and by Cor. 4.5 it follows that (4.60) is a homotopy equivalence, as desired. \square

Definition 4.31. Let \mathcal{A}, \mathcal{B} , and \mathcal{C} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

Define the natural transformations

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}}(M, -), \quad (4.61)$$

$$M^{\bar{\mathcal{A}}} \otimes_{\mathcal{A}} (-) \xrightarrow{\eta_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(M, -) \quad (4.62)$$

of functors $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, respectively, as the compositions

$$(-) \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{(4.30)} \mathrm{Hom}_{\mathcal{B}}(\mathcal{B}, -) \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{B}}(M, \mathcal{B}) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{B}}(M, -),$$

$$\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}) \otimes_{\mathcal{A}} (-) \xrightarrow{(4.30)} \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}) \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, -) \xrightarrow{\mathrm{cmps}} \mathrm{Hom}_{\mathcal{A}}(M, -).$$

Lemma 4.32. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

(1) M is \mathcal{B} -perfect if and only if the map

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}}(M, -)$$

is a homotopy equivalence of functors $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \rightarrow \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ for any DG-category \mathcal{C} .

(2) M is \mathcal{A} -perfect if and only if the map

$$M^{\bar{\mathcal{A}}} \otimes_{\mathcal{A}} (-) \xrightarrow{\eta_{\mathcal{A}}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, -)$$

is a homotopy equivalence of functors $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \rightarrow \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ for any DG-category \mathcal{C} .

Proof. We only give the proof for the assertion (1), as the proof for (2) is identical.

Assume that M is \mathcal{B} -perfect. Since $(-) \xrightarrow{(4.30)} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, -)$ equals the composition

$$(-) \xrightarrow{\mathrm{mlt}} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, (-) \otimes_{\mathcal{B}} \mathcal{B}) \xrightarrow{(4.29) \circ (-)} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, -)$$

it follows that the map

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$$

equals the composition

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{(4.60)} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, (-) \otimes_{\mathcal{B}} \mathcal{B}) \xrightarrow{(4.29) \circ (-)} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, -).$$

The first map in this composition is a homotopy equivalence by Lemma 4.30, and the second one is a homotopy equivalence since (4.29) is. We conclude that $\eta_{\mathcal{B}}$ is also a homotopy equivalence, as desired.

Conversely, assume that

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$$

is a homotopy equivalence on all of $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. By Prop. 4.10 it follows that

$$(-) \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -).$$

Thus $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M, -)$ commutes with infinite direct sums, i.e. M is \mathcal{B} -perfect. \square

Lemma 4.33. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$.*

If M is \mathcal{B} -perfect, there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence

$$N^{\bar{\mathcal{C}}} \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \longrightarrow (M \otimes_{\mathcal{B}} N)^{\bar{\mathcal{C}}}. \quad (4.63)$$

If N is \mathcal{B} -perfect, there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence

$$N^{\bar{\mathcal{B}}} \otimes_{\mathcal{B}} M^{\bar{\mathcal{A}}} \longrightarrow (M \otimes_{\mathcal{B}} N)^{\bar{\mathcal{A}}}. \quad (4.64)$$

Proof. Similarly to the proof of Lemma 2.12 in [AL13], define (4.63) to be the composition

$$N^{\bar{\mathcal{C}}} \otimes_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\eta_{\mathcal{B}}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, N^{\bar{\mathcal{C}}}) \xrightarrow{\mathrm{adjunction}} \overline{\mathrm{Hom}}_{\mathcal{C}}(M \otimes_{\mathcal{B}} N, \mathcal{C}). \quad (4.65)$$

The first composant is a homotopy equivalence by Lemma 4.32 and the second composant is the Tensor-Hom adjunction isomorphism. Thus (4.63) is itself a homotopy equivalence.

We define (4.64) similarly. \square

4.5. Convolution functor for $\mathrm{Pre}\text{-}\mathrm{Tr}(\overline{\mathbf{Mod}})$. Let \mathcal{A} be a DG-category. It is well known that $\mathbf{Mod}\text{-}\mathcal{A}$ is a strongly pretriangulated category, cf. [BK90], [AL13, 3.2]. That is, the natural inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} \hookrightarrow \mathrm{Pre}\text{-}\mathrm{Tr}\mathbf{Mod}\text{-}\mathcal{A}$$

is an equivalence of categories. Its quasi-inverse

$$\mathrm{Pre}\text{-}\mathrm{Tr}\mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{T} \mathbf{Mod}\text{-}\mathcal{A}$$

is the *convolution functor*. Similarly, $\mathbf{Nod}_{\infty}\mathcal{A}$ is strongly pretriangulated and admits a convolution functor.

The category $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is not strongly pretriangulated. It is however *pretriangulated*. In other words,

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \hookrightarrow \mathrm{Pre}\text{-}\mathrm{Tr}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

is only a quasi-equivalence. This is readily seen via the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ of Prop. 4.3. While $\mathbf{Nod}_{\infty}\mathcal{A}$ is strongly pretriangulated, its full subcategory $(\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ is not. This is because when we restrict the convolution functor

$$\mathrm{Pre}\text{-}\mathrm{Tr}\mathbf{Nod}_{\infty}\mathcal{A} \xrightarrow{T_{\infty}} \mathbf{Nod}_{\infty}\mathcal{A}$$

to $\mathrm{Pre}\text{-}\mathrm{Tr}(\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$ its image doesn't restrict to $(\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$. Indeed, let (E_i, α_{ij}) be a twisted complex over $(\mathbf{Nod}_{\infty}\mathcal{A})_{dg}$. That is, E_i are DG \mathcal{A} -modules and α_{ij} are A_{∞} -morphisms between them. Then, $T_{\infty}(E_i, \alpha_{ij})$ is the A_{∞} -module whose underlying DG-module is $T(E_i, \alpha_{ij})$ and whose higher A_{∞} -module structure is defined by the higher operations of the differentials α_{ij} . In particular, this structure will not generally speaking be trivial unless the higher operations are all zero, i.e. unless α_{ij} are regular DG morphisms.

The fact that $T_\infty(\text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{dg})$ doesn't land in $(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ can be readily fixed by applying the semifree resolution $(-)^{\infty}_{\mathcal{A}} \mathcal{A}$ of §3.5. We then obtain a functor

$$\text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{dg} \xrightarrow{T_\infty} (\mathbf{Nod}_\infty \mathcal{A})_{hu} \xrightarrow{(-)^{\infty}_{\mathcal{A}} \mathcal{A}} (\mathbf{Nod}_\infty \mathcal{A})_{dg} \quad (4.66)$$

which is a quasi-equivalence because both of its compositants are. It is also, by construction, a quasi-inverse of the natural inclusion $(\mathbf{Nod}_\infty \mathcal{A})_{dg} \rightarrow \text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{dg}$ and thus an analogue of a convolution functor. In this section, we translate these considerations to the case of $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Throughout this section, we adopt the following notation. Given a morphism $\alpha: E \rightarrow F$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ we denote by $\hat{\alpha}$ the underlying $\mathbf{Mod}\text{-}\mathcal{A}$ morphism $E \otimes \bar{\mathcal{A}} \rightarrow F$.

Recall the natural inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(4.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

of Prop. 4.2. It gets identified under the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty \mathcal{A})_{dg}$ of Prop. 4.3 with the inclusion

$$\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_\infty \mathcal{A})_{dg}^{\text{strict}} \hookrightarrow (\mathbf{Nod}_\infty \mathcal{A})_{dg}.$$

On the other hand, by Cor. 3.5 the semifree resolution

$$(\mathbf{Nod}_\infty \mathcal{A})_{hu} \xrightarrow{(-)^{\infty}_{\mathcal{A}} \mathcal{A}} (\mathbf{Nod}_\infty \mathcal{A})_{dg}$$

factors through $\mathbf{Mod}\text{-}\mathcal{A} = (\mathbf{Nod}_\infty \mathcal{A})_{dg}^{\text{strict}}$. As $(-)^{\infty}_{\mathcal{A}} \mathcal{A}$ is identified with $(-)^{\overline{\otimes}_{\mathcal{A}} \mathcal{A}}$ by the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty \mathcal{A})_{dg}$, it follows that

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{(-)^{\overline{\otimes}_{\mathcal{A}} \mathcal{A}}} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

factors as

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \rightarrow \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(4.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}. \quad (4.67)$$

Definition 4.34. Define the functor

$$\overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\widetilde{(-)}} \mathbf{Mod}\text{-}\mathcal{A} \quad (4.68)$$

to be the factorisation of $(-)^{\overline{\otimes}_{\mathcal{A}} \mathcal{A}}$ given in (4.67).

Lemma 4.35. Let $E \in \overline{\mathbf{Mod}}\text{-}\mathcal{A}$, then

$$\tilde{E} = E \otimes_{\mathcal{A}} \bar{\mathcal{A}}. \quad (4.69)$$

Let $\alpha \in \text{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E, F)$, then

$$\tilde{\alpha} = (\hat{\alpha} \otimes \text{Id}) \circ (\text{Id} \otimes \Delta). \quad (4.70)$$

Proof. Direct verification. \square

Definition 4.36. Define the *convolution functor*

$$\text{Pre-Tr} \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\bar{T}} \overline{\mathbf{Mod}}\text{-}\mathcal{A} \quad (4.71)$$

as the composition

$$\text{Pre-Tr} \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\widetilde{(-)}} \text{Pre-Tr} \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{T} \mathbf{Mod}\text{-}\mathcal{A} \xrightarrow{(4.7)} \overline{\mathbf{Mod}}\text{-}\mathcal{A}. \quad (4.72)$$

Proposition 4.37. The functor \bar{T} gets identified by the isomorphism $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \simeq (\mathbf{Nod}_\infty \mathcal{A})_{dg}$ with the functor

$$\text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{dg} \xrightarrow{(4.66)} (\mathbf{Nod}_\infty \mathcal{A})_{dg}.$$

Proof. In the diagram

$$\begin{array}{ccccc} (\mathbf{Nod}_\infty \mathcal{A})_{hu} & \xrightarrow{(-)^{\infty}_{\mathcal{A}} \mathcal{A}} & \mathbf{Mod}\text{-}\mathcal{A} & & \\ \downarrow & & \downarrow & & \\ \text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{dg} & \hookrightarrow & \text{Pre-Tr}(\mathbf{Nod}_\infty \mathcal{A})_{hu} & \xrightarrow{(-)^{\infty}_{\mathcal{A}} \mathcal{A}} & \text{Pre-Tr} \mathbf{Mod}\text{-}\mathcal{A} \\ \downarrow T_\infty & & \downarrow T & & \downarrow T \\ (\mathbf{Nod}_\infty \mathcal{A})_{hu} & \xrightarrow{(-)^{\infty}_{\mathcal{A}} \mathcal{A}} & \mathbf{Mod}\text{-}\mathcal{A} & \hookrightarrow & (\mathbf{Nod}_\infty \mathcal{A})_{dg} \end{array}$$

the upper rectangle and the outer perimeter of the two rectangles clearly commute. Therefore the lower rectangle commutes as well. The desired assertion now follows by contemplating the diagram. \square

Corollary 4.38. *The convolution functor*

$$\text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A} \xrightarrow{\bar{T}} \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

is a quasi-equivalence and a homotopy inverse of the natural inclusion $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \hookrightarrow \text{Pre-Tr } \overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Let \mathcal{A} and \mathcal{B} be DG-categories. We define the convolution functor

$$\text{Pre-Tr } \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \xrightarrow{\bar{T}} \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$$

similarly.

Lemma 4.39. *Let (E_i, α_{ij}) be a one-sided twisted complex in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and let E be its convolution.*

- (1) *Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and let F be its convolution. Then there is an $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ homotopy equivalence*

$$\left\{ \bigoplus_{k+l=i} E_k \otimes_{\mathcal{B}} F_l, \sum_{l+m=j} (-1)^{l(k-m+1)} \alpha_{km} \otimes \text{Id}_l + \sum_{k+n=j} (-1)^k \text{Id}_k \otimes \beta_{ln} \right\} \rightarrow E \otimes_{\mathcal{B}} F. \quad (4.73)$$

- (2) *Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and let F be its convolution. Then there is a $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ homotopy equivalence*

$$\left\{ \bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\} \rightarrow \text{Hom}_{\mathcal{B}}(E, F). \quad (4.74)$$

- (3) *Let (F_i, β_{ij}) be a one-sided twisted complex in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and let F be its convolution. Then there is a $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ homotopy equivalence*

$$\left\{ \bigoplus_{l-k=i} \text{Hom}_{\mathcal{A}}(E_k, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ \alpha_{mk} + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right\} \rightarrow \text{Hom}_{\mathcal{A}}(E, F). \quad (4.75)$$

Proof. (1): By the definition of the convolution functor as the composition (4.72) it suffices to show that $T \circ (-)$ applied to the twisted complex in the LHS of (4.73) is quasi-isomorphic to $E \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F$ in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$. Since the natural inclusion $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \xrightarrow{(4.7)} \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ maps quasi-isomorphisms to homotopy equivalences, the desired assertion then follows.

By [AL13, Lemma 3.4] we have the following isomorphism in $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$:

$$\left\{ \bigoplus_{k+l=i} \tilde{E}_k \otimes_{\mathcal{B}} \tilde{F}_l, \sum_{l+m=j} (-1)^{l(k-m+1)} \tilde{\alpha}_{km} \otimes \text{Id}_l + \sum_{k+n=j} (-1)^k \text{Id}_k \otimes \tilde{\beta}_{ln} \right\} \simeq E \otimes_{\mathcal{B}} F. \quad (4.76)$$

Let (G_i, γ_{ij}) be the image of the twisted complex in the LHS of (4.73) under $\widetilde{(-)}$. We therefore have

$$G_i = \bigoplus_{k+l=i} \bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}}$$

and thus there is a map from (G_i, γ_{ij}) to the twisted complex in the LHS of (4.76) whose only non-zero components are degree 0 homotopy equivalences

$$\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}} \xrightarrow{\text{Id}^{\otimes 2} \otimes \Delta \otimes \text{Id}^{\otimes 2}} (\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \otimes_{\mathcal{B}} (\bar{\mathcal{B}} \otimes_{\mathcal{B}} F_l \otimes_{\mathcal{C}} \bar{\mathcal{C}}).$$

One readily checks that these commute with the differentials of the twisted complexes, thus the resulting map from $\{G_i, \gamma_{ij}\}$ to the LHS of (4.76) is closed. Therefore by Cor. 2.2 it is a quasi-isomorphism. Thus $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $E \otimes_{\mathcal{B}} F$, and thus to $E \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} F$, as desired.

- (2): Similar to the proof of (1) let (G_i, γ_{ij}) be the image of the twisted complex in the LHS of (4.74) under $\widetilde{(-)}$. It suffices to show that $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $\text{Hom}_{\mathcal{B}}(E \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F)$ in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$.

By [AL13, Lemma 3.4], $\text{Hom}_{\mathcal{B}}(E \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F)$ is isomorphic in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ to the convolution of

$$\left(\bigoplus_{l-k=i} \text{Hom}_{\mathcal{B}}(\tilde{E}_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}, \tilde{F}_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ (\tilde{\alpha}_{mk} \otimes \text{Id}) + \sum_{n-k=j} (-1)^{(l-n+1)k} \tilde{\beta}_{ln} \circ (-) \right). \quad (4.77)$$

We have

$$G_i = \bigoplus_{l-k=i} \bar{\mathcal{C}} \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l) \otimes_{\mathcal{A}} \bar{\mathcal{A}}.$$

Consider therefore the following twisted complex over $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$:

$$\left(\bigoplus_{l-k=i} \mathrm{Hom}_{\mathcal{B}}(\tilde{E}_k \otimes \bar{\mathcal{B}}, F_l), \sum_{l-m=j} (-1)^{m(m-k)+l+1} (-) \circ (\tilde{\alpha}_{mk} \otimes \mathrm{Id}) + \sum_{n-k=j} (-1)^{(l-n+1)k} \beta_{ln} \circ (-) \right). \quad (4.78)$$

Consider the map from (4.77) to (4.78) whose components are the maps

$$\mathrm{Hom}_{\mathcal{B}}(\bar{\mathcal{A}} \otimes_{\mathcal{A}} E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, \bar{\mathcal{C}} \otimes_{\mathcal{C}} F_l \otimes_{\mathcal{B}} \bar{\mathcal{B}}) \xrightarrow{(\tau \otimes \mathrm{Id} \otimes \tau) \circ (-)} \mathrm{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l)$$

These are homotopy equivalences in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$, and thus by Cor. 2.2 the induced map between the convolutions of (4.77) and (4.78) is a quasi-isomorphism.

On the other hand, consider the map of twisted complexes from (G_i, γ_{ij}) to (4.78) whose components are the maps

$$\bar{\mathcal{C}} \otimes_{\mathcal{C}} \mathrm{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l) \otimes_{\mathcal{A}} \bar{\mathcal{A}} \xrightarrow{\tau \otimes ((-) \circ (\tau \otimes \mathrm{Id} \otimes \tau)) \otimes \tau} \mathrm{Hom}_{\mathcal{B}}(E_k \otimes_{\mathcal{B}} \bar{\mathcal{B}} \otimes_{\mathcal{B}} \bar{\mathcal{B}}, F_l).$$

Likewise, these are homotopy equivalences in $\mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and therefore $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to the convolution of (4.78). It is therefore quasi-isomorphic to (4.77). We conclude that $\{G_i, \gamma_{ij}\}$ is quasi-isomorphic to $\mathrm{Hom}_{\mathcal{B}}(E \otimes \bar{\mathcal{B}}, F)$, as desired.

(3): This is proved similarly to the assertion (2). □

Lemma 4.40. *Let (E_i, α_{ij}) be a twisted complex over $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Then there are homotopy equivalences:*

$$\{E_i, \alpha_{ij}\}^{\bar{\mathcal{B}}} \rightarrow \{E_{-i}^{\bar{\mathcal{B}}}, (-1)^{j^2+ij+1} \alpha_{(-i)(-j)}^{\bar{\mathcal{B}}}\}; \quad (4.79)$$

$$\{E_i, \alpha_{ij}\}^{\bar{\mathcal{A}}} \rightarrow \{E_{-i}^{\bar{\mathcal{A}}}, (-1)^{j^2+ij+1} \alpha_{(-i)(-j)}^{\bar{\mathcal{A}}}\}. \quad (4.80)$$

Proof. This is proved similarly to Lemma 4.39. Use [AL13, Lemma 3.5] to write the LHS of (4.80) and (4.79) as twisted complexes over $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$. There are obvious maps from these to the images under $\widetilde{(-)}$ of the RHS twisted complexes in (4.80) and (4.79) whose components are all homotopy equivalences. The claim of the Lemma then follows by Cor. 2.2. □

5. HOMOTOPY ADJUNCTION FOR TENSOR FUNCTORS

Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be DG-categories. In this paper, we frequently consider $\mathcal{A}\text{-}\mathcal{B}$ -, $\mathcal{B}\text{-}\mathcal{C}$ -, etc. bimodules as DG-enhancements of continuous exact functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$, $D(\mathcal{B}) \rightarrow D(\mathcal{C})$, etc. Accordingly, whenever it is convenient we adopt the following “functorial” notation: given $F \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $G \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ we write

$$GF \quad \text{for} \quad F \overline{\otimes}_{\mathcal{B}} G \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}.$$

5.1. Tensor functors. Let \mathcal{A} and \mathcal{B} be DG-categories and let

$$D(\mathcal{A}) \xrightarrow{f} D(\mathcal{B})$$

be a DG-enhanceable functor. Recall that f is said to be *continuous* if it commutes with infinite direct sums. As the following proposition demonstrates, this is equivalent to f being a *tensor functor*, that is — a functor given by tensoring by an $\mathcal{A}\text{-}\mathcal{B}$ -bimodule:

Proposition 5.1. *The following are equivalent:*

- (1) f has a right adjoint $r: D(\mathcal{B}) \rightarrow D(\mathcal{A})$.
- (2) f is continuous.
- (3) f is isomorphic to $H^0((-) \overline{\otimes}_{\mathcal{A}} M)$ for some $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

Proof. The implication (1) \Rightarrow (2) is well-known and straightforward, the implication (2) \Rightarrow (3) follows from [Kel94, §6.4], and the implication (3) \Rightarrow (1) follows since by the Tensor-Hom adjunction the functor $(-) \overline{\otimes}_{\mathcal{A}} M$ has the right adjoint $\overline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$. □

Let f satisfy these equivalent conditions. Fix $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ such that $(-) \overline{\otimes}_{\mathcal{A}} M$ enhances f as above. By Prop. 4.12 the functor $\overline{\mathrm{Hom}}_{\mathcal{B}}(M, -)$ is genuinely adjoint to $(-) \overline{\otimes}_{\mathcal{A}} M$ and thus enhances r . We conclude that any adjoint pair (f, r) of functors $D(\mathcal{B}) \longleftrightarrow D(\mathcal{A})$ with f enhanceable can be enhanced by a pair of genuinely adjoint DG-functors.

We are however, more interested in the case where f has left and right adjoints l and r which are also tensor functors. Note, that r always exists, but might not be a tensor functor, while l may not exist, but when it does — it is automatically a tensor functor. The conditions of the existence of l and of r being a tensor functor are easily stated in terms of the properties of the DG bimodule enhancing f :

Proposition 5.2. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a tensor functor. Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ be any enhancement of f .*

- (1) *The following are equivalent:*
 - (a) *The right adjoint r of f is continuous.*
 - (b) *f restricts to $D_c(\mathcal{A}) \rightarrow D_c(\mathcal{B})$.*
 - (c) *M is \mathcal{B} -perfect.*
 - (d) *$H^0((-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}})$ is the right adjoint of f (see Definition 4.26).*
- (2) *The following are equivalent:*
 - (a) *The left adjoint l of f exists.*
 - (b) *l restricts to $D_c(\mathcal{B}) \rightarrow D_c(\mathcal{A})$.*
 - (c) *M is \mathcal{A} -perfect.*
 - (d) *$H^0((-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}})$ is the left adjoint of f (see Definition 4.26).*

Proof. (1): By the definition of adjunction we have $\mathrm{Hom}_{D(\mathcal{B})}(f(-), \bigoplus_{\infty}(-)) \simeq \mathrm{Hom}_{D(\mathcal{A})}(-, r(\bigoplus_{\infty}(-)))$. Thus if f preserves compact objects r is continuous, i.e. (1a) \Rightarrow (1b). For any $a \in \mathcal{A}$ we have $f({}_a\mathcal{A}) \simeq {}_a M$ in $D(\mathcal{B})$ which shows (1b) \Rightarrow (1c). When M is \mathcal{B} -perfect $M^{\overline{\mathcal{B}}}$ is homotopy right adjoint to M by Prop. 5.7, whence (1c) \Rightarrow (1d). Finally, the implication (1d) \Rightarrow (1a) is trivial since tensor functors are continuous.

(2): If l exists, then, by above, it has to preserve compact objects since f is continuous, so (2a) \Rightarrow (2b). The implications (2b) \Rightarrow (2c) and (2c) \Rightarrow (2d) are proved analogously, and again the implication (2d) \Rightarrow (2a) is trivial. \square

5.2. Homotopy adjunction for tensor functors. Let $f: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ be a continuous functor which has left and right adjoints l and r which are also continuous. By Prop. 5.1 we can enhance f , l , and r by DG bimodules. It is not, to our knowledge, always possible to lift the adjunctions of f , l , and r to genuine adjunctions between the corresponding DG tensor functors. In this section we demonstrate that it is always possible to lift them to homotopy adjunctions in an economical and mutually compatible way.

First, we demonstrate that when M is \mathcal{B} -perfect the functors $((-) \overline{\otimes}_{\mathcal{A}} M, (-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}})$ form a homotopy adjoint pair. That is, there exist maps of bimodules in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ such that the corresponding natural transformations of tensor functors define the unit and the counit of the adjunction in homotopy categories. Similarly, when M is \mathcal{A} -perfect $((-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}}, (-) \overline{\otimes}_{\mathcal{A}} M)$ form a homotopy adjoint pair.

It follows immediately from Lemma 4.32 and the Tensor-Hom adjunction that when M is \mathcal{B} -perfect $((-) \overline{\otimes}_{\mathcal{A}} M, (-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}})$ are homotopy adjoint. Similarly, when M is \mathcal{A} -perfect $((-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}}, (-) \overline{\otimes}_{\mathcal{A}} M^{\overline{\mathcal{A}}\overline{\mathcal{A}}})$ are homotopy adjoint. When M is \mathcal{A} -perfect the natural map $M \rightarrow M^{\overline{\mathcal{A}}\overline{\mathcal{A}}}$ is an isomorphism in $D(\mathcal{A}\text{-}\mathcal{B})$, thus $((-) \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}}, (-) \overline{\otimes}_{\mathcal{A}} M)$ are also homotopy adjoint.

However, the abstract fact of these functors being homotopy adjoint is not enough. We next write down certain natural lifts of adjunctions units and counits involved to the maps in $\overline{\mathbf{Mod}}$ between the DG-bimodules involved. We then compute the homotopies which arise when writing down relations between these maps. Our choice of natural lifts significantly reduces the number of choices involved and thus the number of higher differentials in the explicit computations:

Definition 5.3. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Define the *homotopy trace maps*

$$M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{A}}} \xrightarrow{\mathrm{tr}} \mathcal{A} \quad \text{and} \quad M^{\overline{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\mathrm{tr}} \mathcal{B} \quad (5.1)$$

to be the Tensor-Hom adjunction counits applied to the diagonal bimodules \mathcal{A} and \mathcal{B} , respectively.

To define homotopy action maps and to work with the resulting homotopy adjunctions we need to choose and fix the following data:

Definition 5.4. Let \mathcal{A} and \mathcal{B} be DG-categories.

- (1) For every \mathcal{B} -perfect $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ fix once and for all a homotopy inverse

$$\mathrm{Hom}_{\mathcal{B}}(M, M) \xrightarrow{\zeta_{\mathcal{B}}} M \overline{\otimes}_{\mathcal{B}} M^{\overline{\mathcal{B}}} \quad (5.2)$$

of the evaluation map (4.61). Furthermore, choose and fix

$$\nu_{\mathcal{B}} \in \overline{\mathrm{Hom}}_{\mathcal{A}-\mathcal{A}}^{-1}(\overline{\mathrm{Hom}}_{\mathcal{B}}(M, M)) \quad \text{such that } d\nu_{\mathcal{B}} = \eta_{\mathcal{B}} \circ \zeta_{\mathcal{B}} - \mathrm{Id}. \quad (5.3)$$

(2) For every \mathcal{A} -perfect $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ fix once and for all a homotopy inverse

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\zeta_{\mathcal{A}}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \quad (5.4)$$

of the evaluation map (4.62). Furthermore, choose and fix

$$\nu_{\mathcal{A}} \in \overline{\mathrm{Hom}}_{\mathcal{B}-\mathcal{B}}^{-1}(\overline{\mathrm{Hom}}_{\mathcal{A}}(M, M)) \quad \text{such that } d\nu_{\mathcal{A}} = \eta_{\mathcal{A}} \circ \zeta_{\mathcal{A}} - \mathrm{Id}. \quad (5.5)$$

We now define the homotopy action maps:

Definition 5.5. Let \mathcal{A} and \mathcal{B} be DG-categories. For all \mathcal{B} -perfect (resp. \mathcal{A} -perfect) $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ define the *homotopy \mathcal{A} -action* (resp. *\mathcal{B} -action*) map

$$\mathcal{A} \xrightarrow{\mathrm{act}} M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}}, \quad (5.6)$$

$$\text{resp. } \mathcal{B} \xrightarrow{\mathrm{act}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \quad (5.7)$$

to be the composition

$$\mathcal{A} \xrightarrow{\mathrm{act}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\zeta_{\mathcal{B}}} M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}}, \quad (5.8)$$

$$\text{resp. } \mathcal{B} \xrightarrow{\mathrm{act}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\zeta_{\mathcal{A}}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M. \quad (5.9)$$

Finally, we define the maps whose differentials we prove below to be the difference between our homotopy adjunctions and genuine ones:

Definition 5.6. (1) Define $\chi_{\mathcal{B}}, \chi_{\mathcal{A}} \in \overline{\mathrm{Hom}}_{\mathcal{A}-\mathcal{B}}^{-1}(M, M)$ by the $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ maps

$$M \xrightarrow{\mathrm{act} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, M) \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\nu_{\mathcal{B}} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, M) \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\mathrm{cmps}} M, \quad (5.10)$$

$$M \xrightarrow{\mathrm{act} \otimes \mathrm{Id}} M \overline{\otimes}_{\mathcal{B}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\mathrm{Id} \otimes \nu_{\mathcal{A}}} M \overline{\otimes}_{\mathcal{B}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \xrightarrow{\mathrm{cmps}} M. \quad (5.11)$$

(2) Define $\chi'_{\mathcal{B}}, \xi_{\mathcal{B}} \in \overline{\mathrm{Hom}}_{\mathcal{B}-\mathcal{A}}^{-1}(M^{\bar{\mathcal{B}}}, M^{\bar{\mathcal{B}}})$ and $\chi'_{\mathcal{A}}, \xi_{\mathcal{A}} \in \overline{\mathrm{Hom}}_{\mathcal{B}-\mathcal{A}}^{-1}(M^{\bar{\mathcal{A}}}, M^{\bar{\mathcal{A}}})$ to be the $\mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ maps

$$M^{\bar{\mathcal{B}}} \xrightarrow{\mathrm{Id} \otimes \mathrm{act}} M^{\bar{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\mathrm{Id} \otimes \nu_{\mathcal{B}}} M^{\bar{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} \overline{\mathrm{Hom}}_{\mathcal{B}}(M, M) \xrightarrow{\mathrm{cmps}} M^{\bar{\mathcal{B}}} \quad (5.12)$$

$$M^{\bar{\mathcal{B}}} \xrightarrow{\mathrm{Id} \otimes \mathrm{act}} M^{\bar{\mathcal{B}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\mathrm{cmps} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\kappa_{\mathcal{B}} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{B}}} \xrightarrow{\mathrm{cmps}} M^{\bar{\mathcal{B}}} \quad (5.13)$$

$$M^{\bar{\mathcal{A}}} \xrightarrow{\mathrm{act} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \xrightarrow{\nu_{\mathcal{A}} \otimes \mathrm{Id}} \overline{\mathrm{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \xrightarrow{\mathrm{cmps}} M^{\bar{\mathcal{A}}} \quad (5.14)$$

$$M^{\bar{\mathcal{A}}} \xrightarrow{\mathrm{act} \otimes \mathrm{Id}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \xrightarrow{\mathrm{Id} \otimes \mathrm{cmps}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} \overline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \xrightarrow{\mathrm{Id} \otimes \kappa_{\mathcal{A}}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} \overline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \xrightarrow{\mathrm{cmps}} M^{\bar{\mathcal{A}}} \quad (5.15)$$

Here $\kappa_{\mathcal{A}}$ is the degree -1 map introduced in §4.3 in Definition 4.23 where, in the notation of loc. cit., we set $\mathcal{A} = \mathcal{B}$ and M to be the diagonal bimodule. The map $\kappa_{\mathcal{B}}$ is the analogue for the right action on the diagonal bimodule.

It is now convenient to adopt the functorial notation explained in the beginning of this section. Let F denote the bimodule $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and R and L denote the bimodules $M^{\bar{\mathcal{B}}}$ and $M^{\bar{\mathcal{A}}}$ in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

Proposition 5.7. Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

(1) If M is \mathcal{A} -perfect, we have in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, respectively:

$$F \xrightarrow{\mathrm{act} F} FLF \xrightarrow{F \mathrm{tr}} F = \mathrm{Id} + d\chi_{\mathcal{A}}, \quad (5.16)$$

$$L \xrightarrow{L \mathrm{act}} LFL \xrightarrow{\mathrm{tr} L} L = \mathrm{Id} + d\chi'_{\mathcal{A}} + d\xi_{\mathcal{A}}. \quad (5.17)$$

(2) If M is \mathcal{B} -perfect, we have in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, respectively:

$$F \xrightarrow{F \mathrm{act}} FRF \xrightarrow{\mathrm{tr} F} F = \mathrm{Id} + d\chi_{\mathcal{B}}, \quad (5.18)$$

$$R \xrightarrow{\mathrm{act} R} RFR \xrightarrow{R \mathrm{tr}} R = \mathrm{Id} + d\chi'_{\mathcal{B}} + d\xi_{\mathcal{B}}. \quad (5.19)$$

Proof. We only prove the assertion (1), the assertion (2) is proved similarly.

Consider the following diagram of morphisms in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$:

$$\begin{array}{ccccc}
 M & \xrightarrow{\text{Id} \otimes \text{act}} & M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M & \xrightarrow{\text{tr} \otimes \text{Id}} & \mathcal{A} \overline{\otimes}_{\mathcal{A}} M \\
 & \searrow \text{Id} \otimes \text{act} & \downarrow \text{Id} \otimes \eta_{\mathcal{A}} & \swarrow \text{Id} \otimes (4.30) & \downarrow (4.29) \\
 & & M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) & \xrightarrow{\text{ev}} & M \\
 & & & \nwarrow \text{ev} & \\
 & & & \mathcal{A} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, M) &
 \end{array}$$

(A)

The triangle (A) commutes up to $d(\text{Id} \otimes \nu_{\mathcal{A}})$. The rest of the diagram commutes: the pentagon — by definition of the evaluation map, and the triangle — by direct verification. Therefore the perimeter of the diagram commutes up to $d\xi_{\mathcal{A}}$.

The composition of the lower-left half of the perimeter is readily verified to be the identity morphism. On the other hand, the composition of the upper-right half of the perimeter is the LHS of (5.16). Since the perimeter of the diagram commutes up to $d\chi_{\mathcal{A}}$, the equality in (5.16) follows.

Next, consider the following diagram of morphisms in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$:

$$\begin{array}{ccccc}
 M^{\bar{\mathcal{A}}} & \xrightarrow{\text{act} \otimes \text{Id}} & M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} & \xrightarrow{\text{Id} \otimes \text{tr}} & M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} \\
 & \searrow \text{act} \otimes \text{Id} & \downarrow \eta_{\mathcal{A}} \otimes \text{Id} & \swarrow \text{Id} \otimes (\text{cmps} \circ ((4.30) \otimes \text{Id})) & \downarrow (4.29) \\
 & & \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} & \xrightarrow{\text{cmps}} & M^{\bar{\mathcal{A}}} \\
 & & & \nwarrow \text{cmps} & \\
 & & & M^{\bar{\mathcal{A}}} \overline{\otimes}_{\mathcal{A}} \overline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) &
 \end{array}$$

(A) (B)

The triangle (A) commutes up to $\text{Id} \otimes d\nu_{\mathcal{A}}$ and the triangle (B) commutes up to $\text{cmps} \circ d\kappa_{\mathcal{A}}$. The rest of the diagram commutes: the quadrilateral commutes by the associativity of the composition in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and the triangle commutes by the definition of trace map. It follows that the perimeter commutes up to $d\chi'_{\mathcal{A}} + d\xi_{\mathcal{A}}$, whence (5.17). \square

Corollary 5.8. *Let \mathcal{A} and \mathcal{B} be DG-categories and let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.*

(1) *If M is \mathcal{A} -perfect, we have in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, respectively:*

$$LF \xrightarrow{L \text{ act } F} LFLF \xrightarrow{\text{tr } LF - LF \text{ tr}} LF = d(\xi_{\mathcal{A}} F), \quad (5.20)$$

$$FL \xrightarrow{FL \text{ act} - \text{act } FL} FLFL \xrightarrow{F \text{ tr } L} FL = d(F\xi_{\mathcal{A}}). \quad (5.21)$$

(2) *If M is \mathcal{B} -perfect, we have in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, respectively:*

$$FR \xrightarrow{F \text{ act } R} FRFR \xrightarrow{FR \text{ tr} - \text{tr } FR} FR = d(F\xi_{\mathcal{B}}), \quad (5.22)$$

$$RF \xrightarrow{\text{act } RF - RF \text{ act}} RFRF \xrightarrow{R \text{ tr } F} RF = d(\xi_{\mathcal{B}} F). \quad (5.23)$$

Proof. By Prop. 5.7 the LHS of (5.20) equals $d(\chi'_{\mathcal{A}} F + \xi_{\mathcal{A}} F - L\chi_{\mathcal{A}})$, and $\chi'_{\mathcal{A}} F - L\chi_{\mathcal{A}}$ is the composition

$$M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \rightarrow M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \xrightarrow{\text{Id} \otimes \nu_{\mathcal{A}} \otimes \text{Id} - \text{Id} \otimes \nu_{\mathcal{A}} \otimes \text{Id}} M \overline{\otimes}_{\mathcal{B}} \overline{\text{Hom}}_{\mathcal{A}}(M, M) \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}} \rightarrow M \overline{\otimes}_{\mathcal{B}} M^{\bar{\mathcal{A}}}$$

which is 0. The other assertions work out similarly. \square

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